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MEASURING ANISOTROPY OF MAGNETIC SUSCEPTIBILITY ON A SLOWLY SPINNING SPECIMEN - BASIC THEORY

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1. STATIC AND SPINNING SPECIMEN MEASUREMENT OF THE SUSCEPTIBILITY

In most magnetic susceptibility bridges the static specimen measurement method is used. In the specimen a system of *n* directions is defined, which is called the design of measurement. The measurement consists of *n* elementary steps. In the *m*-th step the specimen is inserted in the measuring coil in such a way so that the *m*-th direction may coincide with the measuring coil axis. The measuring design cannot be chosen arbitrarily. It is necessary, but not sufficient that the design may have at least six directions. In practice a bigger number is used for several reasons, e.g. 15 in the bridge KLY-2, or in a bridge KLY-3 in manual mode when using the program SUSAM.

In the spinning specimen method the position of the specimen is also changed, but continuously instead of discrete steps. The specimen is rotated about 3 axes perpendicular to one another.

The output signal is harmonic, composed from sine and cosine components. Its frequency is twice the frequency of rotation. From this signal the anisotropy can be computed, but the measurement must be completed with one measurement of bulk susceptibility.

In case of the static specimen measurement the mathematical background is less complicated. The bridge need not any exacting mechanical parts. Although the manipulation with specimen is easy, it is time consumpting and requires many manual operations. On the other hand, the method of spinning specimen needs considerably more complicated mathematical methods, which is of course not the problem of the user. It also requires an exacting mechanism for inserting the specimen and spinning it. The measuring procedure is substantially more rapid and the need for manual manipulation is reduced to minimum. But the most important advantage consists in fact, that the sensitivity as well as the accuracy of computed main directions are substantially higher than those of static specimen method.

2. DIRECTIONAL SUSCEPTIBILITY

Regardless of the method used, in every position of the specimen, so called directional susceptibility is measured that corresponds to that direction in the specimen which coincides with the coil axis. Since the directional susceptibility represents the basic notion of the theory of measurement, the more

detailed explanation is needed.

Let us have a specimen S (Fig. 1) in a magnetic field $\underline{H}$ whose direction is $\underline{d}$. The induced magnetization will be denoted $\underline{M}$. In an anisotropic matter in common case the direction of $\underline{M}$ will differ from the directions of $\underline{H}$ or $\underline{d}$.

Now we project orthogonally the vector $\underline{M}$ to the direction of $\underline{H}$ or $\underline{d}$ and denote the length of projection M_D. The ratio

(1) $$æ_D = M_D / H$$

is the directional susceptibility corresponding to the direction $\underline{d}$.

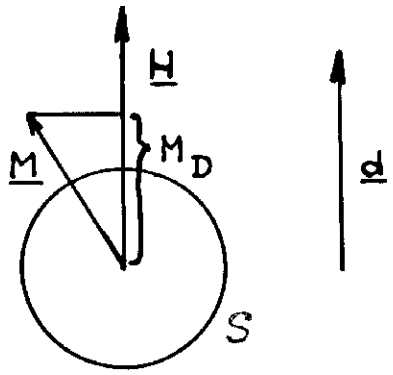

Fig 1. To the definition of the directional susceptibility

3.SUSCEPTIBILITY AND DEVIATORIC SUSCEPTIBILITY

For the subsequent considerations the specimen cordinate system is used. It is a cartesian coordinate system whose axes x_1, x_2, x_3 are connected with some well-defined directions on the specimen. (In case of a cubic specimen are used the directions determined by the edges of it.)

From the physical point of view, the magnetic susceptibility is a symmetrical tensor of second rank. It can be equivalently expressed by a square matrix 3 by 3. Due to this possibility one may write

(2) $$\underline{k} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

where $k_{ij} = k_{ji}$ for $i, j = 1, 2, 3$

Here $\underline{k}$ is the (matrix representation of) susceptibility tensor which is also called full succeptibility tensor.

From the rotational measurement it cannot be determined the full tensor $\underline{k}$, but only s.c. deviatoric tensor $\underline{q}$ which is defined in following way

(3) $$\underline{q} = \underline{k} - æ\underline{I}$$

where æ is the mean susceptibility, $æ = (k_{11} + k_{22} + k_{33}) / 3$ and

$\underline{I}$ is the identity matrix which has in principal diagonal 1's and in non-diagonal positions 0's. It is obvious, that for the "mean" deviatoric susceptibility holds

(4) $$\beta = (g_{11} + g_{22} + g_{33}) / 3 = 0$$

4. PRINCIPAL SUSCEPTIBILITIES AND PRINCIPAL DIRECTIONS

Let us denote the eigenvalues of the matrix $\underline{k}$ through $æ_1$, $æ_2$, $æ_3$ and the respective eigenvectors through $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$. These quantities satisfy, as well known, the relation

(5) $$\underline{k}\underline{p}_i = æ\, \underline{p}_i \qquad (i = 1, 2, 3)$$

The unit vectors $\underline{p}_i$ are called the principal directions, the respective numbers $æ_i$ principal susceptibilities. Formal reasons will make us choose a numbering of quantities that $æ_1 \geq æ_2 \geq æ_3$. Then the principal susceptibilities in order considered are called maximum, intermediate and minimum susceptibility.

The matrix of deviatoric susceptibility $\underline{g}$ can be treated in the same way. Thus the principal deviatoric suceptibilities β_1, β_2, β_3 are got and the same principal directions as in the preceding case.

The principal directions $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$ define a new coordinate system which is called the principal coordinate system and has axes x_1^P, x_2^P, x_3^P. It will be discussed later.

4. THE ELEMENTARY METHOD OF COMPUTING THE DEVIATORIC TENSOR FROM THE THREE ROTATIONS

The simplification in this section is based on fact that for the time being all statistical considerations will be left out. Let us imagine rotation of cylindrical specimen S (which is for determinateness considered as cylindrical) about the axis x_3 that is oriented upwards, see Fig. 2.

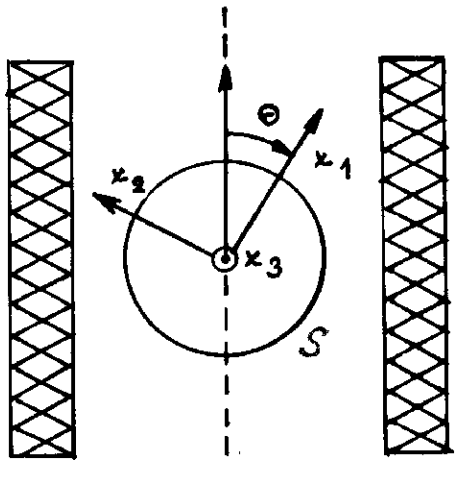

Fig. 2 Spinning of a cylindrical specimen

Using the above definition, one may write for the instantaneous directional susceptibility

$$(3)\quad \text{æ}_D^{(3)} = [\cos\theta \quad \sin\theta \quad 0]\cdot\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}\cdot\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

The superfix indicates, that the directional susceptibility is related to the axis x_3.

After a simple algebra we obtain

$$(4)\quad \text{æ}_D^{(3)} = \tfrac{1}{2}(k_{11} + k_{22}) + \tfrac{1}{2}(k_{11} - k_{22})\cos 2\theta + k_{12}\sin 2\theta$$

and by cyclical permutation of indices

$$\text{æ}_D^{(1)} = \tfrac{1}{2}(k_{22} + k_{33}) + \tfrac{1}{2}(k_{22} - k_{33})\cos 2\theta + k_{23}\sin 2\theta$$

$$\text{æ}_D^{(2)} = \tfrac{1}{2}(k_{33} + k_{11}) + \tfrac{1}{2}(k_{33} - k_{11})\cos 2\theta + k_{31}\sin 2\theta$$

Note that if the specimen rotates with a constant angular speed making f r.p.s, the directional susceptibilities are harmonical functions of t with the frequency $2f$. In practice usually very low frequency is used, say of the order 0.1 Hz.

By measuring three curves are obtained that are characterized by cosine coefficients a_i and sine coefficients b_i. The constant "DC" components are not considered.

$$(6)\qquad y^{(3)} = a_3\cos 2\theta + b_3\sin 2\theta$$
$$y^{(1)} = a_1\cos 2\theta + b_1\sin 2\theta$$
$$y^{(2)} = a_2\cos 2\theta + b_2\sin 2\theta$$

For determining coefficients one half-revolution would do, but usually a bigger number m of half-revolutions is used and the results are averaged.

Comparison of the of the cosine and sine components in Eqs (5) and (6) with concomitant replacing symbols k_{ij} by symbols g_{ij} yields

$$g_{12} = b_3, \quad g_{23} = b_1, \quad g_{31} = b_2 \tag{7}$$

$$\begin{aligned} g_{11} - g_{22} &= 2a_3 \\ g_{22} - g_{33} &= 2a_1 \\ -g_{11} + g_{33} &= 2a_2 \end{aligned} \tag{8}$$

The Eqs (8) represent directly the solution for non-diagonal elements of the computed deviatoric tensor, but the solution of the Eqs (8) is not so easy as it may appear. He who is familiar with elementary linear algebra sees at a glance that the system of Eqs (8) has a singular matrix. We shall assume, that the quantities a_1, a_2, a_3 do not equal zero simultaneously. Then the solution exists if and only if

$$a_1 + a_2 + a_3 = 0 \tag{9}$$

But the solution is not unique, there is an infinite number of solutions. In this situation we - quite naturally - assume that the condition (4) is met. In this case the solution is unique.

The complete solution for g_{ij} can be written in a matrix form as follows

$$(10)\quad \begin{bmatrix} g_{11} \\ g_{22} \\ g_{33} \\ g_{12} \\ g_{23} \\ g_{31} \end{bmatrix} = \begin{bmatrix} & -2/3 & 2/3 & & & & \\ 2/3 & & -2/3 & & & & \\ -2/3 & 2/3 & & & & & \\ & & & & & 1 & \\ & & & 1 & & & \\ & & & & 1 & & \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The above smart mathematical theory is, unfortunately, of little practical use. In measurement, the condition (9) is always fullfiled only approximately which is the consequence of measuring errors. So we must search a compromise solution which is in certain sense optimum. For such a solution the Least Squares method is used as a rule. Strange may it appear, but this method yields precisely the result (10). When applying the Least Squares method, we usually simultanously estimate the errors of measurement. In this case this estimate would be useless since the task has one degree of freedom only. (Six components are measured and five independent parameters are determined). The estimation of errors would thus be extremaly poor. How to avoid this principal disadvantage will be explained in the next section.

5. LEAST SQUARES METHOD WITH INCREASED DEGREES OF FREEDOM

The basic idea is very simple. When rotating about certain axis we can divide the complete rotation formally in N partial measurements (each consisting of m half-revolutions). If e.g. number $N = 2$ ($N = 4$) then for cosine and sine components we obtain 12 (24) values; since we determine 5 independent elements of the tensor, the problem has 7 (19) degrees of freedom. The experience shows that 7 degrees of freedom is sufficient for estimating errors, so we restrict ourselves for simplicity's sake to $N = 2$.

(11)

Measured components		Averages
$z_1 = a_{11}$	}	a_1
$z_2 = a_{12}$		
$z_3 = a_{21}$	}	a_2
$z_4 = a_{22}$		
$z_5 = a_{31}$	}	a_3
$z_6 = a_{32}$		
$z_7 = b_{11}$	}	b_1
$z_8 = b_{12}$		
$z_9 = b_{21}$	}	b_2
$z_{10} = b_{22}$		
$z_{11} = b_{31}$	}	b_3
$z_{12} = b_{32}$		

Symbols z_i, $i = 1$ to 12 are introduced for the numbering of searched tensor components to be simple.

The Least Squares solution is again described by the Eq. (10) without any changes.

5.1 BASIC STANDARD ERROR AND ITS ESTIMATE

We are now standing at the very threshold of the statical problems. Such considerations are rather complex and lie far beyond the scope of this work. Nevertheless, it can be useful to give a very brief overlook of the problem so that the user may understand the results of the program better.

We are entitled to assume that all measured components z_1, z_2, ..., z_{12} have the same standard error σ and are independent one to another. The quantity σ will be called the basic standard error of the problem considered, its statistical estimate will be denoted s.

Let S_o be so called residual sum of squares,

$$S_o = S - R \tag{12}$$

where S is sum of squares of the 12 measured values,

$$S = \sum_{i=1}^{12} z^2 \tag{13}$$

and R is so called reduction in sum of squares

$$R = N \left[\frac{3}{2} \left(\hat{g}_{11}^2 + \hat{g}_{22}^2 + \hat{g}_{11}\hat{g}_{22} \right) + \hat{g}_{12}^2 + \hat{g}_{23}^2 + \hat{g}_{31}^2 \right] \tag{14}$$

The estimate s of basic standard error will now read

$$s = \sqrt{\frac{S_o}{6N - 5}} \tag{15}$$

Remark: The carets (ˆ) in Eq. (14) emphasize that the respective quantities are merely the estimates of the respective true quantities. This denotation is necessary in statistical considerations.

5.2 COVARIANCE MATRIX OF THE COMPONENTS OF TENSOR

The tensor of deviatoric susceptibility being symmetrical can alternatively be expressed by a six components vector

(16)

$$\underline{g} = [g_{11}, g_{22}, g_{33}, g_{12}, g_{23}, g_{31}]' = [g_1, g_2, g_3, g_4, g_5, g_6]'$$

The respective estimates will be denoted in similar way using carets.

Note that for certain statistical considerations also only a five elements vector can be used. This is enabled by the fact that the three main components of the tensor are controlled by the linear condition (9). This method, however, will not be adopted here.

For the random variables $\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_6$ the covariance matrix can be written as follows

(17)

$$\sigma^2\underline{V} = \sigma^2 \begin{bmatrix} V_{11}, V_{12}, \ldots, V_{16} \\ V_{21}, V_{22}, \ldots, V_{26} \\ \ldots\ldots\ldots\ldots\ldots \\ V_{61}, V_{62}, \ldots, V_{66} \end{bmatrix}$$

In the above equation V_{ij} is the covariance of the random variables $\hat{g}_i$ and $\hat{g}_j$ ($i, j = 1, 2, \ldots, 6$). The symbol σ^2 is the quadrate of basic standard error (the basic dispersion), and $\underline{V}$ is certain matrix of constants which is called the skeleton of covariance matrix.

Under practical conditions we have no knowledge on the true covariance matrix $\underline{V}$. We are, therefore, only able to draw on the corresponding estimate $\hat{\underline{V}}$

$$\hat{\underline{V}} = s^2\underline{V} \tag{17}$$

where s^2 is the quadrate of the estimate of the basic standard error (the estimate of basic dispersion). The matrix $\underline{V}$ is assumed be the same as in Eq.(16).

6. TENSOR IN PRINCIPAL COORDINATE SYSTEM

For statistical considerations is very useful to express the tensor in the principal coordinate system with the axes x^P_1, x^P_2, x^P_3 defined by the triad of vectors $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$ or in the estimated principal coordinate system defined by the triad of vectors $\hat{\underline{p}}_1$, $\hat{\underline{p}}_2$, $\hat{\underline{p}}_3$. Which system is used is obvious from context as a rule.

The tensor components will be denoted g^P_{ij} ($i, j = 1, 2, 3$) or g^P_k ($k = 1, 2, \ldots, 6$), the respective estimates $\hat{g}^P_{ij}$ and $\hat{g}^P_k$.

The covariance matrix in principal coordinate system will read

(18) $$\sigma^2 \underline{V}^P$$

Transforming $\underline{V}$ to $\underline{V}^P$ is rather complicated task, nevertheless the respective computer program works very fast. Note that in some special cases - when the method is "rotatable" - the skeleton matrix $\underline{V}$ is invariant with respect to the transformation of coordinates. This is e.g. the case of the static specimen method of mesurement in 15 directions.

7. TESTS OF ANISOTROPY

7.1 THREE-DIMENSION ANISOTROPY TEST

This test serves to verify whether the differences between the the estimated principal susceptibilities compared to measuring errors are great enough for us to be entitled to consider the specimen as anisotropic.

Two hypotheses will be considered:

H_o - that the specimen is isotropic (zero hypothesis) and

H_1 - that the specimen is anisotropic (alternative hypothesis).

Under the zero hypothesis H_o, the random variable $Q = R / \sigma^2$, where R is given by Eq. (14), has χ^2 (chi-squared) distribution on 5 degrees of freedom. Furthermore, the expression

s^2 / σ^2 - regardless to H_o - has the distribution χ^2 / m, which is independent on Q. The constant $m = 6N - 5$ and in the case considered $m = 7$.

As a consequence, the statistic

(20) $$F = (Q / 5) / (s^2 / \sigma^2) = R / (5s^2)$$

has F-distribution on 5 and m degrees of freedom. In the case considered $m = 7$.

Let us consider the $(100 - \sigma)$ % quantile of F-distribution on 5 and m degrees of freedom. It will be denoted $F_{5,\ m;\ (1 - \alpha)}$. In case

(21) $$F > F_{5,\ m,\ (1 - \alpha)}$$

we shall reject the hypothesis H_o (that the specimen is isotropic) in favour of H_1 (that the specimen is anisotropic) on the level of significance of $(100 - \alpha)$ % . In the opposite case we are not entitled to reject H_o in favour of H_1 on the given level of significance.

The above formulation is exact from the mathematical point of view, but, unfortunately, not easy to understand. Therefore we shall try to give an explanation in more lucid way. Let us make an arrangement that the level of significance will be 95 % as it is usual in most rock magnetic works. When we measure many times a perfectly isotropic specimen and follow the above instruction for each individual case then in 95 % cases we shall come to the correct conclusion that the specimen is isotropic, but in the remaining 5 % cases we shall make the false conclusion that the

specimen is anisotropic. This false conclusion is called the error of first kind. When, on the other hand, we measure weakly anisotropic specimen, we shall make false conclusion from time to time that the specimen is isotropic. This is the error of second kind. The evaluation of the probability of the second kind error needs very refined statistical methods that cannot be discussed here.

Note that the 95 % quantile

$$F_{5,\,7;\,95} = 3.9715$$

The essential purpose of the F-test is thus to eliminate the cases of "apparent" anisotropy that is the consequence of measuring errors only. According to the value of statistic F it can be be judged the "quality" of suceptibility measurement. One must realize that the statistic F is *quadratically* dependent on this intuitive "quality". No wonder that in measuring strongly anisotropic specimens very great values are obtained of the order of thousands and more.

7.1 TEST OF ROTATIONAL ANISOTROPY

In practical application we often come across a case of an approximately rotational anisotropy. There are two types of such an anistropy. In the first type

(22a) $$æ_1 \approx æ_2 \quad \text{while} \quad æ_3 \ll æ_1, æ_2 \quad \text{i.e.}$$

$$ß_1 \approx ß_2 \qquad ß_3 \ll ß_1, ß_2$$

(This type is very frequent in sediments where the axis x_3^P is perpendicular to the bedding planes.) In the opposite case

(22b) $\quad \text{æ}_2 \approx \text{æ}_3$ while $\text{æ}_1 \gg \text{æ}_2, \text{æ}_3$ i.e.

$\beta_2 \approx \beta_3 \qquad \beta_1 \gg \beta_2, \beta_3$

To be brief we shall discuss the first type only, the second type being the precise analogy of the first one.

The test considered serves to verify whether the difference between the measured principal susceptibilities $\hat{\beta}_1$ and $\hat{\beta}_2$ compared to measuring errors are great enough for us to be entitled to consider the specimen as anisotropic in the plane (x_1^P, x_2^P).

The zero hypothesis H_o will now be the statement that the specimen is isotropic in the plane (x_1^P, x_2^P), i.e. has rotational symmetry with respect to the axis x_3^P, and $\beta_1 = \beta_2$. The alternative hypothesis H_1 will be the statement, that the specimen is anisotropic in the plane (x_1, x_2), i.e. is triaxially anisotropic, and $\beta_1 =/= \beta_2$.

The statistic for testing will be

(23) $$F = Z_3 \, (\hat{\beta}_1 - \hat{\beta}_2) \, / \, (2s^2)$$

where Z_3 is certain constant which can be computed from the skeleton $\underline{V}^P$ of the covariance matrix in the principal co-ordinate system. Deriving of Z_3 is easy but rather tedious therefore it will be omitted. Under the zero hypothesis H_o, the

statistic F has the F-distribution on 2 and m degrees of freedom.

Consider $(100 - \alpha)$ % quantile $F_{2,\, m,\, (1 - \alpha)}$ of the F-distribution on 2 and m degrees of freedom. If

$$F > F_{2,\, m,\, (100 - \alpha)} \tag{24}$$

the zero hypothesis H_o will be rejected in favour of the alternative hypothesis H_1 at the level of significance $(100 - \alpha)$ %. If the inequality (24) is false the zero hypothesis H_o cannot be rejected at the level of significance $(100 - \alpha)$ %.

In the problem under discussion $m = 7$. The conventional level of significance will be chosen, viz. 95 %. Thus the numerical value of the quantile will be

$$F_{2,\, 7;\, 95} = 4.7374$$

The practical explanation of the result of the test is quite analogous to that in section 7.1.

8. CONFIDENCE REGIONS AND ANGLES OF PRINCIPAL DIRECTIONS

Due to the measuring errors the principal directions cannot be determined accurately. Fortunately, we can give a good statistical estimates of their reliability - the confidence

regions and confidence angles. For better intelligibility's sake the third principal directions will be considered. The reader can re-write the results for the first and second directions easily.

8.1 ASSOCIATED PROBLEM OF PROBABILITY CALCULUS

To begin with, the problem will be discussed from the point of view of probability calculus - it will be assumed, that the actual principal susceptibilities and principal directions are known.

In Fig. 3 the problem of variation of the 3rd principal direction is depicted. In certain measurement the direction $\underline{\tilde{p}}_3$ will differ from the true direction $\underline{p}_3$ by the vector $d\underline{p}_3$ lying in the plane π_3, which is perpendicular to $\underline{p}_3$ and whose components in directions $\underline{p}_1$ and $\underline{p}_2$ are dp_1 and dp_2 respectively. One can get easily

$$dp_{13} = dg^{P}_{31} \; / \; (\beta_3 - \beta_1) \tag{25}$$

$$dp_{23} = dg^{P}_{23} \; / \; (\beta_3 - \beta_2)$$

$$d\underline{p}_3 = dg^{P}_{31}\,\underline{p}_1 \; / \; (\beta_3 - \beta_1) + dg^{P}_{23}\,\underline{p}_2 \; / \; (\beta_3 - \beta_2)$$

To make the equations easier to survey, the index "3" in the rightmost position will be omitted wherever possible. Using this convention instead of dp_{13} and dp_{23} we shall rather write dp_1 and dp_2 etc.

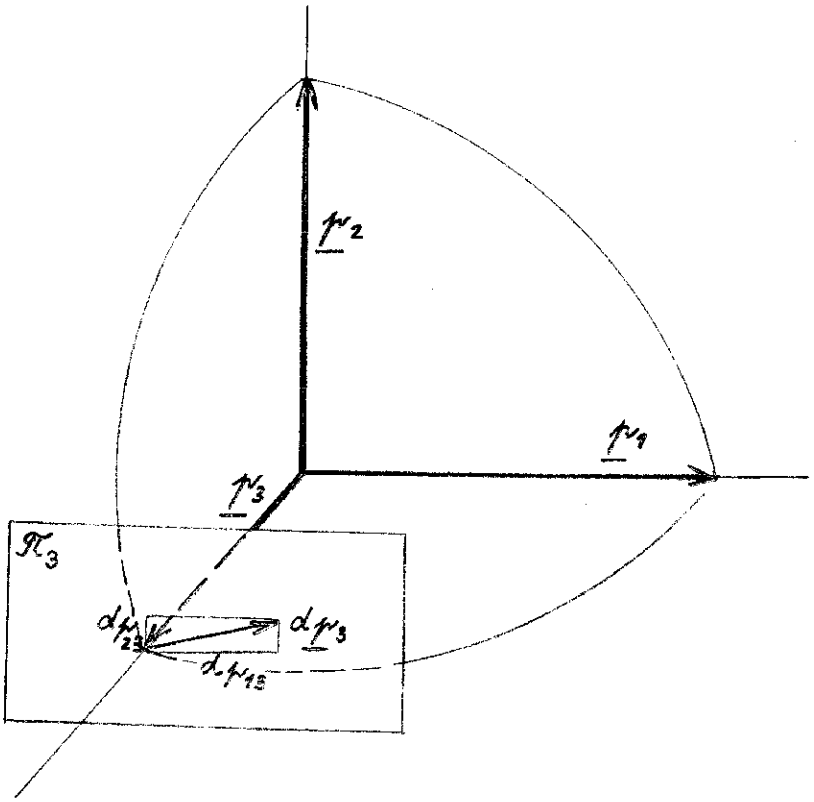

Fig. 3 Variations on principal vector $\underline{p}_3$

From the covariance matrix $\sigma^2\underline{V}^P$ using Eqs(25) one can find the covariance matrix of the components dp_{13} and dp_{23},

$$(26) \qquad \sigma^2\underline{W} = \begin{bmatrix} \dfrac{V^P_{44}}{(\beta_3 - \beta_1)^2} & \dfrac{V^P_{65}}{(\beta_3 - \beta_1)(\beta_3 - \beta_2)} \\ \dfrac{V^P_{65}}{(\beta_3 - \beta_1)(\beta_3 - \beta_2)} & \dfrac{V^P_{55}}{(\beta_3 - \beta_2)^2} \end{bmatrix}$$

The quadratic form

$$(27) \qquad Q^2 = [dp_1 \; dp_2]' \; \underline{W}^{-1} \begin{bmatrix} dp_1 \\ dp_2 \end{bmatrix}$$

has χ^2-distribution on two degrees of freedom. Drawing on Eq. (25) an ellipse E_3 can be constructed which will eliminate $(100 - \alpha)$ % of the least probable positions of the vector $\underline{p}_3$. This ellipse is called tolerance ellipse and can be defined by the equation

$$(28) \qquad Q^2 = \chi^2_{2;(100-\alpha)}$$

Denote the eigenvalues of the matrix $\underline{W}$ (not $\underline{W}^{-1}$)

$$\delta_1 \quad \text{and} \quad \delta_2$$

the respective eigenvectors

$$[c_{11} \quad c_{21}]' \quad \text{and} \quad [c_{12} \quad c_{22}]'$$

The lenghts of the semi-axes of the ellipse E_3 will be

$$(29) \qquad e_{T1} = \sigma\delta_1 \chi^2_{2;(100-\alpha)} \qquad e_{T2} = \sigma\delta_2 \chi^2_{2;(100-\alpha)}$$

and will be located along the directions

$$(30) \qquad c_{11}\underline{p}_1 + c_{21}\underline{p}_2 \quad \text{and} \quad c_{12}\underline{p}_1 + c_{22}\underline{p}_2$$

The ellipse E_3 will be projected centrally onto a unit sphere; in this way we arrive at the region in which $(100 - \alpha)$% of the terminating points of the estimated vectors $\hat{\underline{p}}_3$ are located. The

projection of tolerance ellipse is called tolerance region. The lenghts of the projections of the axes of the tolerance ellipse

(31) $$e_{13} = \tan^{-1} e_{T1} \quad \text{and} \quad e_{23} = \tan^{-1} e_{T2}$$

taken as arches are called tolerance angles. The index "3" has been introduced again to emphasize that these tolerance angles deal with the third principal direction.

8.2 STATISTICAL PROBLEM

Under practical condition we have knowledge neither of the actual tensor $\underline{g}^P$ nor of the covariance matrix $\sigma^2\underline{W}^P$. Therefore it is necessary to draw on the respective estimates

$$\hat{\underline{g}}^P \quad \text{and} \quad \overline{\sigma^2\underline{W}}$$

It will be assumed that the only source of variations in the estimate of covariance matrix is the variation of the estimate of basic dispersion s^2, while the skeleton $\underline{W}$ is fixed so that one can write

(32) $$\overline{\sigma^2\underline{W}} = s^2\underline{W}$$

The quadratic form

$$F' = [dp_1 \; dp_2]' \; (1 / s^2) \; \underline{W}^{-1} \begin{bmatrix} dp_1 \\ dp_2 \end{bmatrix} \tag{33}$$

unlike to the form in Eq. (27), has not the χ^2-distribution. A formal rearangement of Eq. (31) yields

$$F' = [dp_1 \; dp_2]' \; (1 / \sigma^2) \; \underline{W}^{-1} \begin{bmatrix} dp_1 \\ dp_2 \end{bmatrix} \; : \; \frac{s^2}{\sigma^2}$$

The dividend has obviously χ^2-distribution on 2 degrees of freedom, while the divisor has χ^2-distribution on 7 degrees of freedom divided by 7. Thus the statistic

$$F = F' \; / \; 2 \tag{33}$$

has F-distribution on 2 and 7 degrees of freedom.

The further procedure will be similar to that in Section 8.1. The ellipse $\hat{E}_3$ lying in the plane $\hat{\pi}_3$ given by the equation

$$F = F_{2,\,7;\,(100-\alpha)}$$

will be consructed and called the confidence ellipse of the third principal direction p_3 on the level of significance (100 - α) %. The eigenvectors of the matrix $\hat{\underline{W}}$ determine the directions of the semi-axes of $\hat{E}_3$. Let δ_1 and δ_2 be the eigenvalues of matrix $\hat{\underline{W}}$ (not $\hat{\underline{W}}^{-1}$). Then the lenghts of semi-axes will be

(35)
$$e_{T13} = s\left(2\delta\, F_{2,\,7,\,(100-\alpha)}\right)^{\frac{1}{2}}$$

$$e_{T23} = s\left(2\delta\, F_{2,\,7,\,(100-\alpha)}\right)^{\frac{1}{2}}$$

The index "3" has just been introduced again. The central projection of $\hat{E}$ onto a unit sphere is the confidence region $\hat{R}$. The lengths of projection of semi-axes of ellipse $\hat{E}_3$, taken as angles, are

(36)
$$e_{13} = \tan^{-1} e_{T13} \quad \text{and} \quad e_{23} = \tan^{-1} e_{T23}$$

and are called confidence angles of the third direction.

Note that the respective results for the first and second principal direction can be obtained simply by the cyclical permutations of indices.

The meaning of the confidence region is not so lucid as that of tolerance region. To begin with, it is essential to emphasize that the tolerance region is a fixed object while the confidence region is a *random* object. That is why it is denoted by a letter with a caret (^).

When measuring the same specimen many times we can expect that the actual mean direction will lie in (100 - α) % within the confidence region and in α % without it. Since the confidence region is a *random* object one ought to say that the actual principal direction lies in the confidence region with certain *confidence* rather then *probability*.

In one measurement of a specimen, three confidence regions are

obtained, one for each principal direction, as it is shown in the Fig. 4 which is a sketch of typical result.

It is obvious, that the three confidence regions are not independent on one another. The 1st "semi-axis" of the 1st (2nd, 3rd) confidence region lies almost precisely on a common great circle with the 2nd "semi-axis" of the 2nd (3rd, 1st) confidence region. At the same time the confidence regions meet the conditions

$$(37) \qquad e_{11} \approx e_{22}\,, \qquad e_{12} \approx e_{23}\,, \qquad e_{13} \approx e_{21}$$

(Sometimes the 1st and the 2nd confidence angle for certain principal direction must be perhaps interchanged). All the six confidence angles are printed by the program, but it seems that even a triad of angles would do.

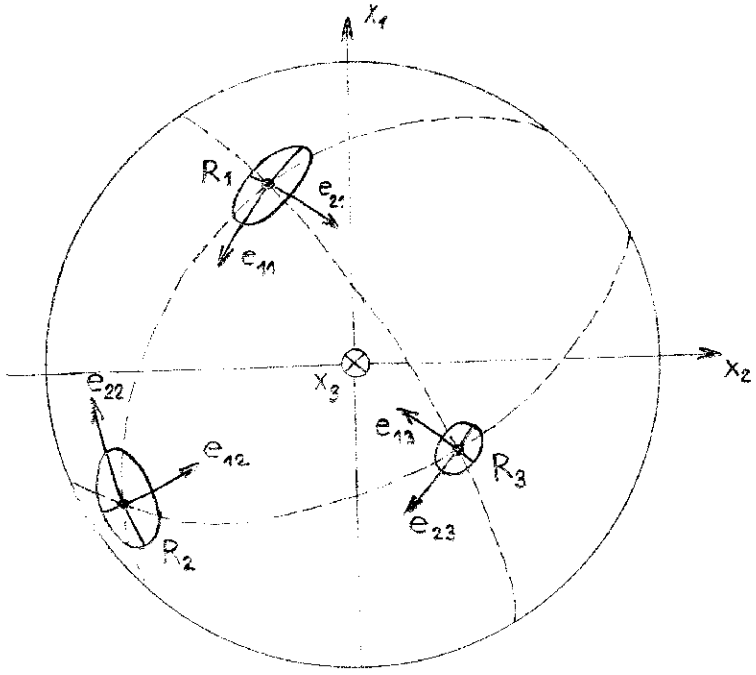

Fig. 4 Example of confidence regions

9. COMPLETION OF DEVIATORIC TENSOR TO THE FULL ONE

When measuring the anisotropy of susceptibility by the spinning specimen method, only the deviatoric tensor of susceptibility $\underline{g}$ is obtained. To get the full tensor one directional susceptibility is needed. Usualy the direction of one axis of the specimen coordinate system is chosen; here will be considered the direction of x_1. The respective direction susceptibility is k_{11}. The full tensor $\underline{k}$ will be obtained if to all the three diagonal elements of the deviatoric tensor $\underline{g}$ will be added the term

(37) $$k_{11} - g_{11}$$

which can be also written in the matrix way as follows

(38) $$\underline{k} = (k_{11} - g_{11})\ \underline{I} + \underline{g}$$

where $\underline{I}$ is the identity matrix.

10. ON THE PROGRAM SUSAR

For measuring the susceptibility by the spinning specimen metod on the KLY-3 susceptibility bridge a rather extensive program SUSAR was written. The source code is in Microsoft Basic language and is accessible to every user of the KLY-3.

The comprehensive part of the program serves for control of the instrument and for maintaining the proper sequence of

operation. The numerical part is not very complicated but it needs to pay utmost attention to the indices which can be confused easily. The crucial point of the numerical problems is finding the eigevalues and eigenvectors of a symmetric matrix. The Jacobi iterative method was chosed that gives eigenvalues and eigenvectors at the same time and ensures perfect orthogonality of the latters. Besides the problem described here the program also performs transformation to further coordinate systems - geographical, paleogeographical and tectonic. For detail see AGICO Print No. 12.

Any practical experience with the program will be greatly appreciated by the present author.

REMARKS ON NOTATION

The notation had to be matched to the editor used, as a consequence there are some minor differences in present document from the notation which is now being settled.

(i) Column matrices (algebraic vectors) are denoted by underscored italics (e.g. $\underline{p}$) - instead of semibold italics.

(ii) All other types of matrices are denoted by upright underscored letters (e.g. $\underline{\mathrm{k}}$) - instead of upright semibold letters.

(iii) The eigenvalues of the deviatoric susceptibility matrix are denoted β - more logical would be γ . #