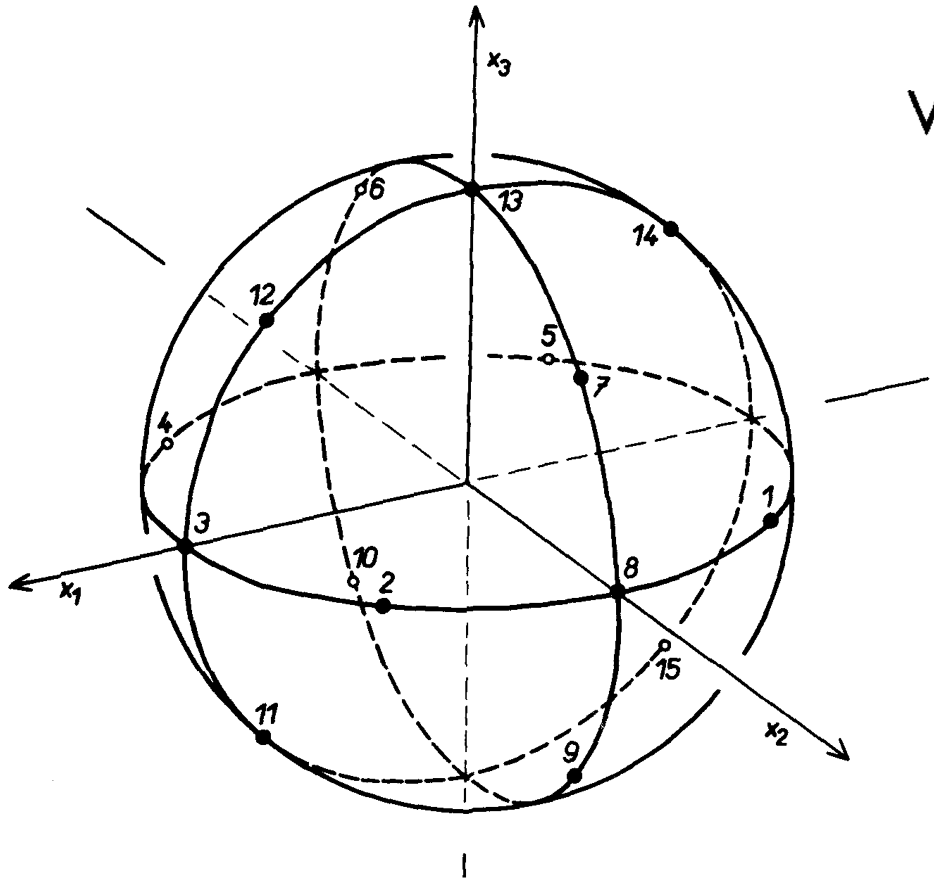
Vít Jelínek



The Statistical Theory of Measuring Anisotropy of Magnetic Susceptibility of Rocks and Its Application

Geofyzika Brno

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THE STATISTICAL THEORY OF MEASURING ANISOTROPY OF MAGNETIC SUSCEPTIBILITY OF ROCKS AND ITS APPLICATION

GEOFYZIKA
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PREFACE

Some time ago, a precision A.C. bridge for measuring magnetic susceptibility of rocks and its anisotropy was developed [13] now produced by the GEOFYZIKA, n.p. under the trade mark KAPPABRIDGE KLY 1. This work resulted from the requirements of the users of the bridge in question, yet we believe it can be of some use to wide circles of workers in the field of rock magnetism.

The theory of measurement of the magnetic susceptibility anisotropy interpreted here originates from the methods of mathematical statistics, in the first place from Hext's statistics of second-order tensor. A description of computing the susceptibility tensor, the principal susceptibilities and principal directions from the measured directional susceptibilities is given, and a new design of measuring directions is proposed, some of whose properties are of a great advantage. Special attention is paid to the question of estimating the accuracy of the computed principal susceptibilities and principal directions, as well as to the question of statistical tests for anisotropy. The theory is extended by a discussion of the two-dimensional problem concerning the rocks whose susceptibility shows an approximately rotational symmetry.

To make possible the application of the principles presented, a computing program ANISO 10 in the language FORTRAN IV was worked out (authors V. Jelínek and M. Franková). The program complexly processes the data measured on a rock specimen : it determines the susceptibility tensor and the parameters derived therefrom in several coordinate systems, carries out tests for anisotropy and computes the accuracy estimates of the results. The description of the program's function is given in this work, the program listing can be obtained from GEOFYZIKA, n.p.

The author would like to thank Mrs. Skorkovská for her operative help in the final redaction, and Dr. F. Hrouda for the experimental material that has been made use of for the illustration.

Brno 1976 Vít Jelínek

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INTRODUCTION

Magnetic susceptibility of a linear anisotropic medium is a second-order tensor quantity. That is why its measurement is in many aspects more complicated than the measurement of the vector or scalar quantities.

The measuring principles used are manifold. The most frequent methods are based on measuring the so called directional susceptibilities, corresponding to certain suitably chosen directions in the rock specimen. (A typical device that performs its measuring on this principle is the A.C. bridge [10, 11, 13].) From the directional susceptibilities the tensor components are then computed. As the susceptibility tensor is symmetrical and thus has six independent components, it would be sufficient to measure directional susceptibilities in six suitably chosen directions. But, usually, a larger number of directions is chosen (e.g. nine, fifteen or eighteen) to reach higher accuracy, to be able to estimate measuring errors and to exclude the measurements laden with gross errors.

The choice of a suitable system of measuring directions represents an interesting problem in itself. From the point of view of the measuring practice an easy realization of the directions is important. From the standpoint of the statistical processing of the results, it is advantageous for the design to be rotatable. The problems of rotatable design are dealt with by Hext [7]; but the designs proposed by him do not satisfy the postulate of an easy realizability.

From the susceptibility tensor the principal susceptibilities and principal directions are computed. From the mathematical point of view the question here is the determination of eigenvalues and eigenvectors of a square matrix that expresses susceptibility tensor.

After the above mentioned computations, we get the susceptibility parameters expressed in the coordinate system associated with the specimen. Nevertheless, the interpretation of the results usually needs the susceptibility parameters to be transformed into further systems - the geographic and tectonic ones.

The most difficult problem is the statistical consideration of the accuracy of the results, especially of the accuracy of the principal susceptibilities and directions. One of the first to try to solve this problem was Stone [12]. But his experiment is based on too simplified conceptions, and the practical applicability of the proposed method is, in our opinion, rather problematic.

A mathematically founded theory was created by Hext [7]. His work deals with the second-order tensor statistics in general, but the motive for its elaboration were the very problems from the field of measuring the anisotropy of magnetic susceptibility.

The conception of his work is very general, and its form rather close. This makes a practical application of it rather difficult. As far as we know from published works, Hext's very fertile ideas were taken advantage of in one case only [10], and then only as a demonstration of measurement processing.

In this work we are trying to interprete the application of the statistics of the second-order tensor in a more intelligible way, to extend it in a certain line, and to suggest the limits of its applicability. Nevertheless, this intention is not our only aim; the work includes all the problems of measuring the anisotropy of magnetic susceptibility in the above mentioned extent. Its logical accomplishment is the computing program that represents an effective tool for transmitting the achieved results to the immediate geophysical practice.

COMMENTS TO SYMBOLICS

We shall briefly state the main principles according to which the vectors, tensors and matrices will be denoted.

All the coordinate systems that will serve to express vectors and tensors will be cartesian. As to this circumstance, no further statement will be made.

A vector (in a physical or geometrical sense of the word) will be denoted by a once underlined letter, e.g. $\underline{d}$. The same denotation will also be used for the column matrix expressing the respective vector in the coordinate system considered as fundamental. In this sense we then write $\underline{d} = [d_1 \; d_2 \; d_3]'$.

The second-order tensor will be denoted by a twice underlined letter, e.g. $\underline{\underline{k}}$. The same symbol will also denote the square matrix expressing the tensor in question in the fundamental coordinate system.

The coordinate system of the specimen that is characterized in chapter 3, will be considered as fundamental. Only in chapter 2 shall we somewhat deviate from this principle.

A matrix expressing a vector or tensor in another than fundamental system will be denoted by the basic symbol completed by a distinguishing index in the top righthand corner, e.g. $\underline{d}^Y$, $\underline{\underline{k}}^Y$.

A once underlined letter will also be the denotation of any column matrix, even if it does not express a vector; a twice underlined letter will denote any matrix that is not a column matrix.

1. THREE-DIMENSIONAL PROBLEM

1.1 MAGNETIC SUSCEPTIBILITY OF ANISOTROPIC MEDIUM, BASIC NOTIONS

Magnetic susceptibility of a linear magnetic medium may be described by a symmetrical second-order tensor $\underline{\underline{k}}$; this tensor will be expressed by a square matrix

$$(1.1) \qquad \underline{\underline{k}} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix} ,$$

or, alternatively, by a column matrix

$$(1.2) \qquad \underline{k} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{bmatrix}' = \\ = \begin{bmatrix} k_{11} & k_{22} & k_{33} & k_{12} & k_{23} & k_{13} \end{bmatrix}'$$

with six components.

Let us choose in the considered linear magnetic medium an arbitrary direction and let this direction be expressed by the unit directional vector $\underline{d} = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}'$.

The quantity

(1.3) $$\varkappa_D = \underline{d}' \underline{\underline{k}}\, \underline{d}$$

will be called directional susceptibility corresponding to the considered direction $\underline{d}$. The relation (1.3) can be modified to

(1.4) $$\varkappa_D = a'(\underline{d})\ \underline{k}\ ,$$

where

(1.5) $$a'(\underline{d}) = \left[d_1^2 \quad d_2^2 \quad d_3^2 \quad 2d_1d_2 \quad 2d_2d_3 \quad 2d_1d_3\right] \ .$$

Let us denote the eigenvalues of the matrix $\underline{\underline{k}}$ through $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, the respective eigenvectors through $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$. These quantities satisfy, as well known, the relation

(1.6) $$\underline{\underline{k}}\ \underline{p}_i = \varkappa_i\ \underline{p}_i \qquad (i = 1, 2, 3)$$

or

(1.7) $$\underline{\underline{p}}' \underline{\underline{k}}\ \underline{\underline{p}} = \begin{bmatrix} \varkappa_1 & & \\ & \varkappa_2 & \\ & & \varkappa_3 \end{bmatrix} ,$$

where $\underline{\underline{p}} = [\underline{p}_1, \ \underline{p}_2, \ \underline{p}_3]$ and $\underline{p}_i = [p_{1i} \ \ p_{2i} \ \ p_{3i}]'$. The vectors $\underline{p}_i$ are called the principal directional vectors of the principal directions, the numbers $\varkappa_i$ principal susceptibilities. Formal reasons will make us choose a numbering of the characteristic quantities that $\varkappa_1 \geqq \varkappa_2 \geqq \varkappa_3$.

If all the three principal susceptibilities are positive, as usual in rocks, the susceptibility tensor may be interpreted geometrically by an ellipsoid called susceptibility ellipsoid [17].

In a general case the principal susceptibilities are distinct. The susceptibility ellipsoid is tri-axial, its semi-axes lie in the directions $\pm \underline{p}_1$, $\pm \underline{p}_2$, $\pm \underline{p}_3$ and their length is $1/\sqrt{\varkappa_1}$, $1/\sqrt{\varkappa_2}$, $1/\sqrt{\varkappa_3}$. The medium in question is called tri-axial anisotropic.

If two principal susceptibilities coincide and the third one is distinct, the susceptibility ellipsoid is rotational. The medium is called rotationally anisotropic.

Finally, a case may occur of all the three principal susceptibilities coinciding. This is a case of an isotropic medium, the susceptibility ellipsoid degenerates into a sphere. The isotropic medium may be considered as a special case of the anisotropic medium.

An important characteristic of the anisotropic medium is the mean susceptibility $\varkappa$, defined as the mean value of all the directional susceptibilities. It is also equal to the arithmetical mean of the principal susceptibilities

(1.8) $$\varkappa = \frac{1}{3}(\varkappa_1 + \varkappa_2 + \varkappa_3) ,$$

or, more generally, to the arithmetical mean of three directional susceptibilities corresponding to three arbitrary, mutually perpendicular directions; it is, further, equal to the arithmetical mean of the diagonal elements of the matrix expressing the susceptibility tensor

(1.9) $$\varkappa = \frac{1}{3}(k_{11} + k_{22} + k_{33}),$$

1.2 ESTIMATING SUSCEPTIBILITY TENSOR FROM THE MEASURED SYSTEM OF DIRECTIONAL SUSCEPTIBILITIES

Let us suppose such a way of measuring, where the device indicates, directly or after a simple correction, the directional susceptibility.

To this type belong the devices [10, 11] as well as the KAPPABRIDGE KLY-1, the prototype of which is described in [13].

We shall start from a situation, where the system of directional susceptibilities has been measured in certain chosen directions.

For the determination of six independent elements of the susceptibility tensor it would be sufficient to measure directional susceptibilities in six suitably chosen directions, from which no two directions are equivalent. Because of reasons stated in the introduction, the measuring is performed in a more extensive system of directions, and the elements of the susceptibility tensor are then determined by means of the least square method [1]. Its application is, in the given case, relatively easy, as the defining equations are linear.

Let us now consider a measurement in n directions $\underline{d}_1$, $\underline{d}_2$, ..., $\underline{d}_n$; the corresponding measured directional susceptibilities will be denoted $\tilde{\varkappa}_{D1}, \tilde{\varkappa}_{D2}, \ldots, \tilde{\varkappa}_{Dn}$. The defining equations will be written in in a matrix form

$$\underline{\tilde{\varkappa}}_D = \underline{\underline{A}}\underline{k} + \underline{\delta} , \qquad (1.10)$$

----

1) The theoretical principles of the least square method can be found in [1], concrete applications to anisotropy in [7, 10, 15].

where $\underline{\varkappa}_D$ and $\underline{\delta}$ are matrices of the type (n x 1) with the components $\varkappa_{Di}$ and δ_i, respectively, n $\geqq$ 6. The matrix $\underline{\underline{A}}$ of the type (n x 6)

$$\underline{\underline{A}} = \begin{bmatrix} \underline{a}'(\underline{d}_1) \\ \cdot \\ \cdot \\ \cdot \\ \underline{a}'(\underline{d}_n) \end{bmatrix} \tag{1.11}$$

will be called - according to Hext - design matrix. The row submatrix $\underline{a}(\underline{d}_i)$ is given by the equation (1.5). We assume that the directions are chosen suitably,so that $\underline{\underline{A}}$ is a full-rank matrix, its rank being 6.

The errors δ_i are all supposed to have the distribution $N(0,\sigma^2)$ and to be mutually independent.For the matrix $\underline{k}$ expressing the susceptibility tensor we shall obtain, by means of the least square method, the best unbiased estimate

$$\hat{\underline{k}} = (\underline{\underline{A}}' \underline{\underline{A}})^{-1} \underline{\underline{A}}' \tilde{\underline{\varkappa}}_D \; . \tag{1.12}$$

The elements of the matrix $\hat{\underline{k}}$ (estimates of the independent components of susceptibility tensor) have normal distribution with the covariance matrix $\sigma^2\underline{\underline{V}}$,

$$\sigma^2\underline{\underline{V}} = \sigma^2(\underline{\underline{A}}' \underline{\underline{A}})^{-1} \; . \tag{1.13}$$

For each of the chosen directions a certain "fitted" value of directional susceptibility can be computed from the estimated susceptibility tensor. These fitted values are given by an equation analogous to (1.10)

(1.14) $$\hat{\underline{\varkappa}}_D = \underline{\underline{A}}\, \hat{\underline{k}}$$

The difference $\tilde{\varkappa}_{Di} - \hat{\varkappa}_{Di}$ is the residual error of the i-th directional susceptibility. The residual sum of squares

(1.15) $$S_o = \left[\tilde{\underline{\varkappa}}_D - \hat{\underline{\varkappa}}_D\right]' \left[\tilde{\underline{\varkappa}}_D - \hat{\underline{\varkappa}}_D\right] =$$

$$= \tilde{\underline{\varkappa}}'_D \tilde{\underline{\varkappa}}_D - \hat{\underline{k}}' \underline{\underline{A}}' \underline{\underline{A}}\, \hat{\underline{k}} =$$

$$= \tilde{\underline{\varkappa}}'_D \tilde{\underline{\varkappa}}_D - \hat{\underline{\varkappa}}'_D \hat{\underline{\varkappa}}_D ,$$

is distributed as $\sigma^2 \chi^2$, where χ^2 is a random variable following chi-square distribution on n - 6 degrees of freedom. The quantity

(1.16) $$s^2 = S_o/(n - 6)$$

represents the unbiased estimation of the dispersion σ^2 .

The dispersion σ^2 and the corresponding standard deviation will be called dispersion of directional susceptibility and standard error of directional susceptibility, respectively [1]).

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[1]) In connection with the computing program the quantities σ^2 and σ will be briefly called fundamental dispersion and fundamental standard deviation, respectively.

1.3 A ROTATABLE DESIGN OF FIFTEEN MEASURING DIRECTIONS

Formally, the choice of measuring directions is limited only by the condition that the design matrix $\underline{\underline{A}}$ must have full rank. That does not mean, of course, that an arbitrary design of directions meeting the stated condition is suitable for practical measurement.

Hext [7] referring to [8] shows that there exist particularly advantageous designs giving covariance matrices that are invariant with respect to any orthonormal transformation (rotation) of the coordinate system.

For a design of directions with the mentioned property the denotation rotatable design is used. Every rotatable design of n directions has the covariant matrix $\sigma^2\underline{\underline{V}}$ of the form

$$\sigma^2\underline{\underline{V}} = \frac{\sigma^2}{4n}\begin{bmatrix} 24 & -6 & -6 & & & \\ -6 & 24 & -6 & & & \\ -6 & -6 & 24 & & & \\ & & & 15 & & \\ & & & & 15 & \\ & & & & & 15 \end{bmatrix} \tag{1.17}$$

and, vice versa, any design of n directions with the covariance matrix of the form stated is rotatable. Hext presents a whole series of rotatable designs that have a common disadvantage in a difficult realization of the individual directions when measuring.

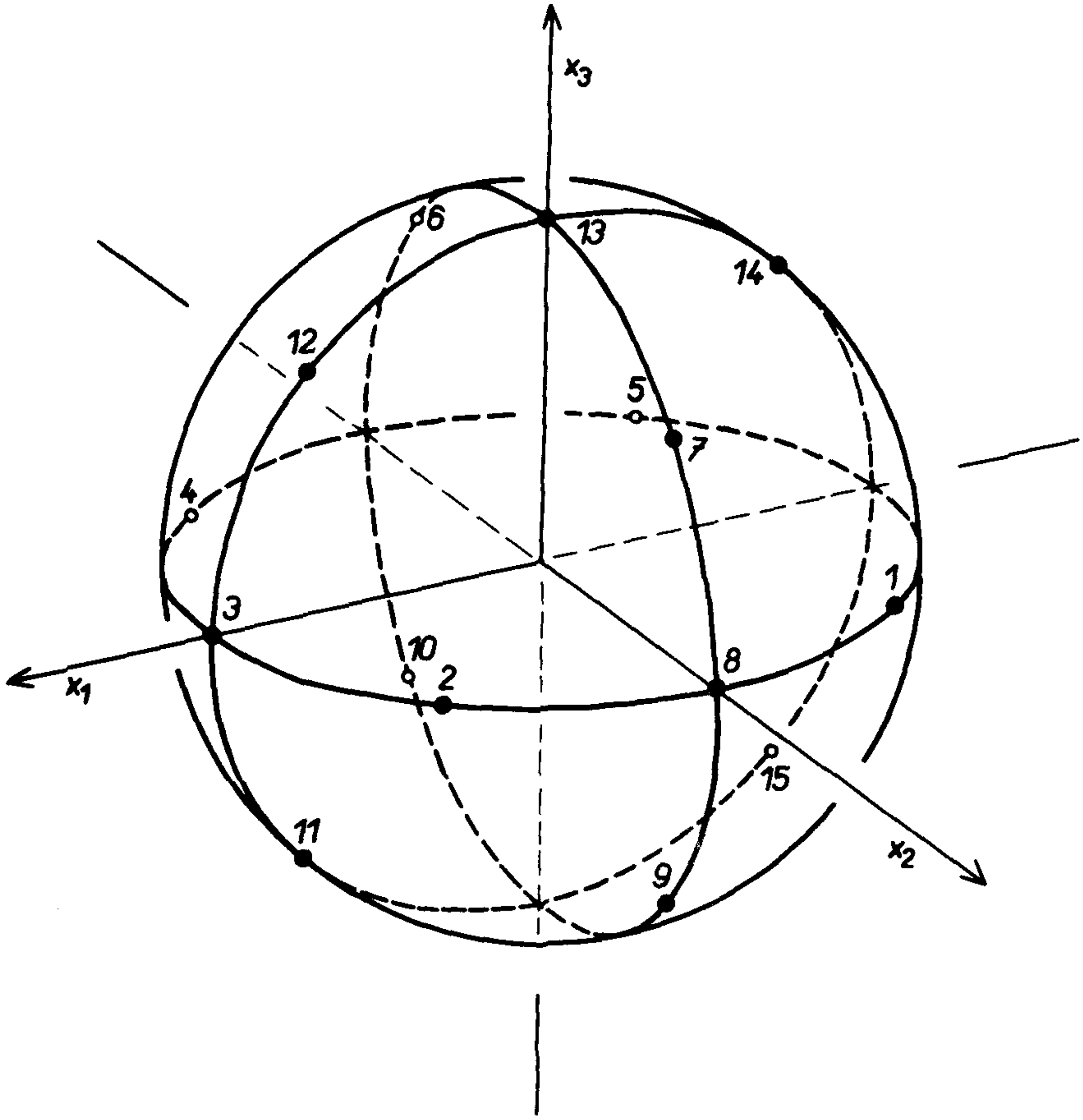

Fig. 1.1 Rotatable design of 15 measuring directions.

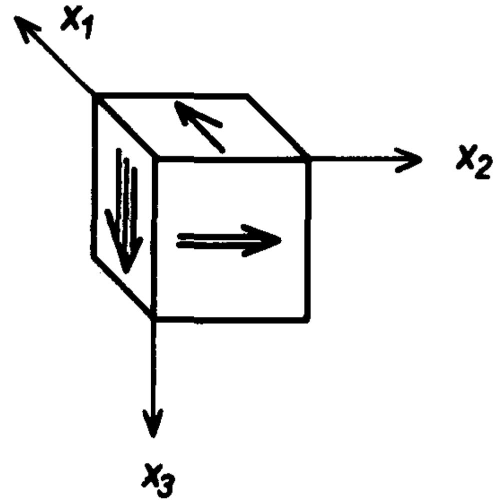

Fig. 1. 2 Marking a cube-shaped specimen for anisotropy measurement.

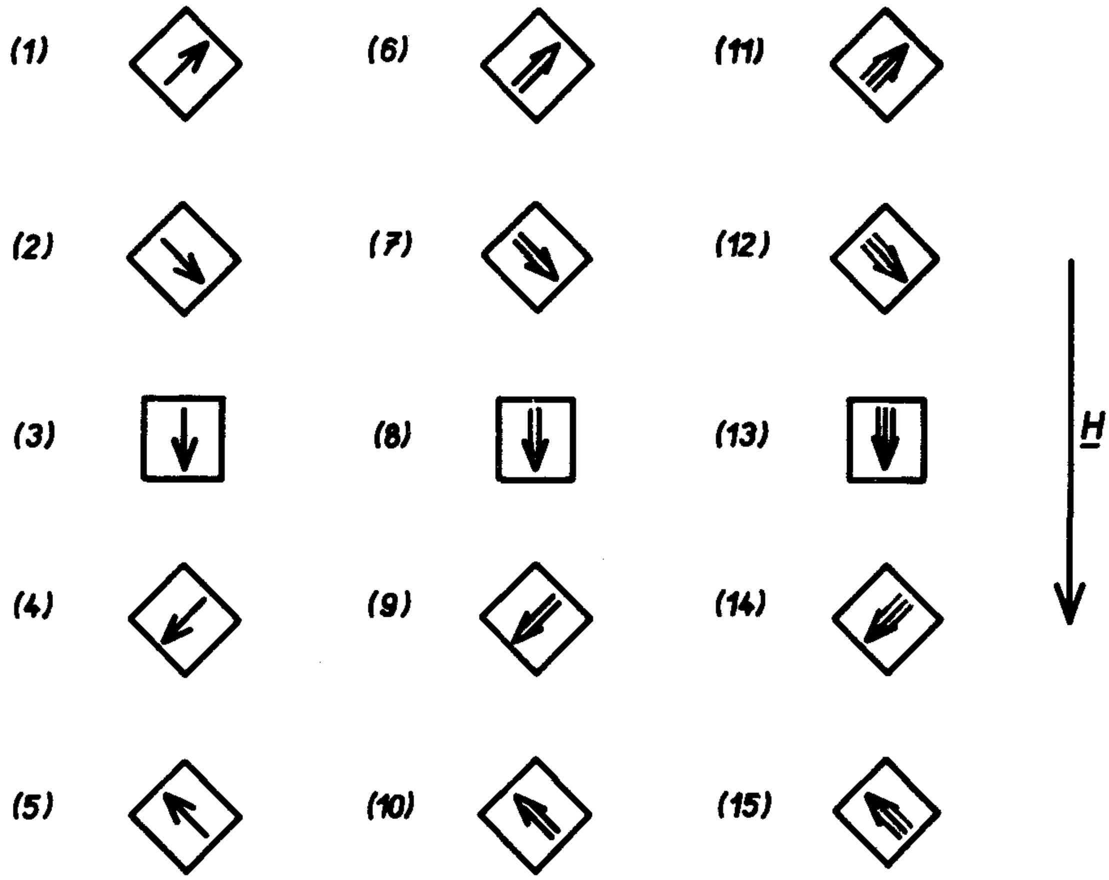

Fig. 1. 3 Implementation of the rotatable design of 15 measuring directions. The positions of a cube shaped specimen with respect to the magnetic field vector $\underline{H}$, i. e. with respect to the directed coil axis, are shown.

We have found a rotatable design of 15 directions, devoid of this disadvantage. This design is illustrated in the figure 1.1 by the end-points of the directional vectors on the unit sphere. It has arisen from the formerly used non-rotatable design of 18 measuring directions by omitting the directions opposite to the directions marked in the figure by the numbers 3, 8, 13. The realization of this rotatable design of 15 directions for a cubic specimen is demonstrated in the figure 1.3; the way of marking the specimen by a triad of arrows is shown in the figure 1.2.

The design matrix $\underline{\underline{A}}$ of the considered rotatable design has the form

(1.18)

$$\underline{\underline{A}} = \frac{1}{2}\begin{bmatrix} 1 & 1 & & -2 & & \\ 1 & 1 & & 2 & & \\ 2 & & & & & \\ 1 & 1 & & -2 & & \\ 1 & 1 & & 2 & & \\ & 1 & 1 & & -2 & \\ & 1 & 1 & & 2 & \\ & 2 & & & & \\ & 1 & 1 & & -2 & \\ & 1 & 1 & & 2 & \\ 1 & & 1 & & & -2 \\ 1 & & 1 & & & 2 \\ & & 2 & & & \\ 1 & & 1 & & & -2 \\ 1 & & 1 & & & 2 \end{bmatrix}.$$

By substituting from (1.18) into (1.12) we shall get the relation for the best unbiased estimate of the components of the susceptibility tensor

(1.19)

$$\begin{bmatrix} \hat{k}_1 \\ \cdot \\ \cdot \\ \cdot \\ \\ \hat{k}_6 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 3 & 3 & 8 & 3 & 3 & -2 & -2 & -2 & -2 & -2 & 3 & 3 & -2 & 3 & 3 \\ 3 & 3 & -2 & 3 & 3 & 3 & 3 & 8 & 3 & 3 & -2 & -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 & -2 & 3 & 3 & -2 & 3 & 3 & 3 & 3 & 8 & 3 & 3 \\ -5 & 5 & & -5 & 5 & & & & & & & & & & \\ & & & & & -5 & 5 & & -5 & 5 & & & & & \\ & & & & & & & & & & -5 & 5 & & -5 & 5 \end{bmatrix} \begin{bmatrix} \tilde{\varkappa}_{D1} \\ \cdot \\ \cdot \\ \cdot \\ \\ \tilde{\varkappa}_{D15} \end{bmatrix}.$$

The covariance matrix will be determined according to (1.13) or (1.17). We get

$$(1.20) \qquad \sigma^2 \underline{\underline{V}} = \sigma^2 \begin{bmatrix} 0.4 & -0.1 & -0.1 & & & \\ -0.1 & 0.4 & -0.1 & & & \\ -0.1 & -0.1 & 0.4 & & & \\ & & & 0.25 & & \\ & & & & 0.25 & \\ & & & & & 0.25 \end{bmatrix} .$$

The best unbiased estimate $\hat{\varkappa}$ of the mean susceptibility is given by the relation

$$(1.21) \qquad \hat{\varkappa} = \frac{1}{3} (\hat{k}_{11} + \hat{k}_{22} + \hat{k}_{33}) ,$$

which immediately follows from the equation (1.9) and from the Gauss-Markov theorem [1].

Using the relations (1.15, 1.19, 1.21) the residual sum of squares for the rotatable design of fifteen measuring directions can be expressed,

(1.22)

$$S_o = \tilde{\varkappa}_D' \tilde{\varkappa}_D - 2(\hat{k}_1^2 + \hat{k}_2^2 + \hat{k}_3^2 + 2\hat{k}_4^2 + 2\hat{k}_5^2 + 2\hat{k}_6^2) - 9 \hat{\varkappa}^2 .$$

1.4 VARIATIONS OF PRINCIPAL SUSCEPTIBILITIES AND OF PRINCIPAL DIRECTIONS

The estimated components of the susceptibility tensor may be, according to the preceding chapter, written in a matrix way

$$(1.23) \qquad \hat{\underline{k}} = \underline{k} + d\underline{k} ,$$

where the matrix $d\underline{k}$ expressing the variations of the estimate $\hat{\underline{k}}$ has central normal distribution with the covariance matrix $\sigma^2 \underline{\underline{V}}$ according to (1.17).

Let us suppose that the variations of the principal directions and principal susceptibilities may be expressed in a differential form

$$(1.24) \qquad \begin{aligned} \hat{\underline{p}}_i &= \underline{p}_i + d\underline{p}_i , \\ \hat{\varkappa}_i &= \varkappa_i + d\varkappa_i \qquad (i = 1, 2, 3) , \end{aligned}$$

and that thus the elements of the covariant matrix $\sigma^2 \underline{\underline{V}}$ are small to such an extent that all the powers and products of the differentials are negligible. This formulation of the assumption is taken over from Hext [7]. From the mathematical point of view it is sure not to be quite correct. An exact formulation would, however, be very complicated.

Essentially, the meaning of the assumption is the following : differences between any two principal susceptibilities must be substantially greater than the standard measuring error σ.

If the assumption in question is satisfied, so - as has been proved by Hext - it holds in the first place that the vector $d\underline{p}_i$ is perpendicular to the vector $\underline{p}_i$. Thus, the vector $d\underline{p}_i$ lies in the plane ϱ_i, which touches the unit sphere in the endpoint of the vector $\underline{p}_i$. In the plane ϱ_i we shall introduce a cartesian system of coordinates with the origin in the endpoint of the vector $\underline{p}_i$ and with the axes parallel to the directions $\underline{p}_j$, $\underline{p}_k$. The components of the vector $d\underline{p}_i$ in this coordinate system will be denoted dp_{ji}, dp_{ki};

$$d\underline{p}_i = \begin{bmatrix} dp_{ji} & dp_{ki} \end{bmatrix}' \tag{1.25}$$

and it holds $\qquad dp_{jk} = -dp_{kj}$.

For variations of the principal susceptibilities and principal directions it holds

$$\begin{bmatrix} d\,\varkappa_1 \\ d\,\varkappa_2 \\ d\,\varkappa_3 \\ dp_{21} \\ dp_{32} \\ dp_{31} \end{bmatrix} = \begin{bmatrix} \underline{b}'_{11} \\ \underline{b}'_{22} \\ \underline{b}'_{33} \\ \underline{b}'_{12}/(\varkappa_1 - \varkappa_2) \\ \underline{b}'_{23}/(\varkappa_2 - \varkappa_3) \\ \underline{b}'_{13}/(\varkappa_1 - \varkappa_3) \end{bmatrix} \begin{bmatrix} dk_1 \\ dk_2 \\ dk_3 \\ dk_4 \\ dk_5 \\ dk_6 \end{bmatrix} , \tag{1.26}$$

the right side will be written, more briefly, $\underline{\underline{B}}\, d\underline{k}$; at the same time

$$\underline{b}'_{jk} = \underline{b}'_{kj} = \big[p_{1j}p_{1k} \quad p_{2j}p_{2k} \quad p_{3j}p_{3k} \quad (p_{1j}p_{2k} + p_{2j}p_{1k}) \quad (p_{2j}p_{3k} + p_{3j}p_{2k}) \quad (p_{1j}p_{3k} + p_{3j}p_{1k}) \big] .$$

The relation (1.26) represents a linear transformation of random variables. From the theory of linear transformations of the random variables, see e.g. [1, 6], it follows that the covariance matrix $\sigma^2\underline{\underline{W}}$

of the elements of the column matrix on the left side of the equation (1.26) will be computed according to the relation

(1.27) $$\sigma^2 \underline{\underline{W}} = \sigma^2 \underline{\beta}\, \underline{\underline{V}}\, \underline{\beta}' .$$

We want to emphasize that further considerations will concern the rotatable design of 15 measuring directions only. The results for other rotatable designs are very similar, whereas the results for non-rotatable designs are substantially more complicated.

In the considered case, for the covariance matrix $\sigma^2 \underline{\underline{W}}$ we get

(1.28)

$$\sigma^2 \underline{\underline{W}} = \sigma^2 \begin{bmatrix} 0.4 & -0.1 & -0.1 & & & \\ -0.1 & 0.4 & -0.1 & & & \\ -0.1 & -0.1 & 0.4 & & & \\ & & & [2(\varkappa_1 - \varkappa_2)]^{-2} & & \\ & & & & [2(\varkappa_2 - \varkappa_3)]^{-2} & \\ & & & & & [2(\varkappa_1 - \varkappa_3)]^{-2} \end{bmatrix} .$$

From the position of the zero non-diagonal elements of $\sigma^2 \underline{\underline{W}}$ it follows that the variations dp_{jk} of the principal directions are mutually independent and also independent of the variations of the principal susceptibilities.

The principal susceptibilities $\varkappa_i$ $(i = 1, 2, 3)$ have altogether normal distribution with dispersion $0.4\,\sigma^2$.

The statistic $u = (\hat{\varkappa}_i - \varkappa_i)/\sqrt{0.4}\,\sigma$ has standard normal distribution, so that the tolerance interval that will exclude 100α % of the least likely results will be

$$(1.29) \qquad \langle \varkappa_i - \sqrt{0.4}\,\sigma\; u_{(1-\alpha/2)},\;\; \varkappa_i + \sqrt{0.4}\,\sigma\; u_{(1-\alpha/2)} \rangle ,$$

where $u_{(1-\alpha/2)}$ is the $100(1-\alpha/2)\%$ quantile of the standard normal distribution.

The vector $d\underline{p}_i$, expressing the variations of the estimated principal direction $\hat{\underline{p}}_i$, has in the plane ρ_i, defined above, bivariate central normal distribution with the covariance matrix

$$(1.30) \qquad \sigma^2 \underline{\underline{W}}_i = \sigma^2 \begin{bmatrix} [2(\varkappa_i - \varkappa_j)]^{-2} & \\ & [2(\varkappa_i - \varkappa_k)]^{-2} \end{bmatrix} .$$

We shall introduce the quantities

$$(1.31) \qquad \begin{aligned} \sigma_{Tji} &= \sigma / \left[2 | \varkappa_i - \varkappa_j | \right] \\ \sigma_{Tki} &= \sigma / \left[2 | \varkappa_i - \varkappa_k | \right] \end{aligned}$$

and call them error parameters of the i-th principal direction. Then we can write

$$(1.32) \qquad \sigma^2 \underline{\underline{W}}_i = \begin{bmatrix} \sigma^2_{Tji} & \\ & \sigma^2_{Tki} \end{bmatrix} .$$

The expression

$$(1.33) \qquad \left(\frac{dp_{ji}}{\sigma_{Tji}} \right)^2 + \left(\frac{dp_{ki}}{\sigma_{Tki}} \right)^2$$

thus has chi-square distribution on two degrees of freedom. In the plane ρ_i a tolerance ellipse can be constructed that will exclude

$100\alpha\%$ of the least likely endpoints of the vector $\hat{\underline{p}}_i$. The semiaxes of the tolerance ellipse are lying in the directions $\underline{p}_j$, $\underline{p}_k$ and their length is

(1.34)
$$\varepsilon_{Tji} = \sigma_{Tji}\left[\chi^2_{2;(1-\alpha)}\right]^{\frac{1}{2}}$$
$$\varepsilon_{Tki} = \sigma_{Tki}\left[\chi^2_{2;(1-\alpha)}\right]^{\frac{1}{2}},$$

where $\chi^2_{2;(1-\alpha)}$ is the $100(1-\alpha)\%$ quantile of chi-square distribution on two degrees of freedom.

We shall project the tolerance ellipse centrally on the unit sphere and call the thus formed region on the sphere tolerance region of the estimated principal direction $\hat{\underline{p}}_i$.

1.5 CONFIDENCE INTERVALS OF PRINCIPAL SUSCEPTIBILITIES, CONFIDENCE REGIONS OF PRINCIPAL DIRECTIONS

In practice, with the susceptibility tensor $\underline{\underline{k}}$ unknown, we can - when considering the variations of the principal susceptibilities and principal directions - start only from the estimated susceptibility tensor $\hat{\underline{\underline{k}}}$.

The estimate of the covariance matrix $\sigma^2\underline{\underline{W}}$ (1.28) will be the matrix

(1.35)
$$s^2\hat{\underline{\underline{W}}},$$

where s^2 is the estimate of the dispersion σ^2, $\hat{\underline{\underline{W}}}$ is the estimate of the matrix $\underline{\underline{W}}$. We can obtain this estimate $\hat{\underline{\underline{W}}}$ by replacing the principal susceptibilities $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ in the matrix $\underline{\underline{W}}$ by their estimates $\hat{\varkappa}_1$, $\hat{\varkappa}_2$, $\hat{\varkappa}_3$.

We shall consider the estimate s^2 of the dispersion σ^2 as the only source of the variations of the matrix $s^2\underline{\underline{W}}$ while the variations of $\hat{\underline{\underline{W}}}$ in the given approximation will not be considered. As we have assumed the differences between the actual principal susceptibilities $\varkappa_i$ to be sufficiently great in comparison with the standard error σ, the inaccuracy thus arisen will be of no importance, which follows from the character of the matrix $\sigma^2\underline{\underline{W}}$, see (1.28).

The estimate of the dispersion of the principal susceptibility $\hat{\varkappa}_i$ then is $0.4s^2$.

We construct the statistic

$$t = (\hat{\varkappa}_i - \hat{\varkappa}_i)/\sqrt{0.4}s = u/(s/\sigma) , \tag{1.36}$$

where

$$u = (\hat{\varkappa}_i - \varkappa_i)/\sqrt{0.4}\,\sigma \quad .$$

The random variable u has the distribution N(0, 1). The expression s/σ is independent of u and is distributed as $(\chi^2/9)^{\frac{1}{2}}$, where χ^2 has chi-square distribution on 9 degrees of freedom. Thus the statistic t has Student distribution on 9 degrees of freedom. Hence it follows that the $100(1 - \alpha)\%$ confidence interval for the principal susceptibility $\varkappa_i$ is

$$\left\langle \hat{\varkappa}_i - \sqrt{0.4}s\ t_{9;(1-\alpha/2)},\ \hat{\varkappa}_i + \sqrt{0.4}s\ t_{9;(1-\alpha/2)} \right\rangle , \tag{1.37}$$

where $t_{9;(1-\alpha/2)}$ is $100(1 - \alpha/2)\%$ quantile of the Student distribution on 9 degrees of freedom.

Estimates s_{Tji} of the error parameters σ_{Tji} are

(1.38)
$$s_{Tji} = s/\left[2|\hat{\alpha}_i - \hat{\alpha}_j|\right]$$
$$s_{Tki} = s/\left[2|\hat{\alpha}_i - \hat{\alpha}_k|\right] .$$

We shall introduce error angles

(1.39)
$$s_{ji} = \tan^{-1} s_{Tji}$$
$$s_{ki} = \tan^{-1} s_{Tki} .$$

Expression

(1.40)
$$\left(\frac{dp_{ji}}{s_{Tji}}\right)^2 + \left(\frac{dp_{ki}}{s_{Tki}}\right)^2 ,$$

is analogous to the expression (1.33). Under the above mentioned assumption that the estimate s^2 of the dispersion σ^2 is the only source of variations, the equation (1.40), using equations (1.31, 1.38), may be arranged to

(1.41)
$$\left[\left(\frac{dp_{ji}}{\sigma_{Tji}}\right)^2 + \left(\frac{dp_{ki}}{\sigma_{Tki}}\right)^2\right] \Big/ \left(\frac{s}{\sigma}\right).$$

The expression in the brackets has again chi-square distribution on two degrees of freedom and is independent of the expression s/σ . Expression s/σ is distributed as $(\chi^2/9)^{\frac{1}{2}}$, where χ^2 has chi-square distribution on 9 degrees of freedom. Thus the expression (1.40) is distributed as 2F, where F has F-distribution on 2 and 9 degrees of freedom.

On the basis of this result we construct an approximate $100(1 - \alpha)\%$ confidence ellipse for the endpoint of the vector $\underline{p}_i$.

The confidence ellipse lies in the plane $\hat{\rho}_i$ that passes through the endpoint of the vector $\hat{\underline{p}}_i$ and is perpendicular to it. The semiaxes of the confidence ellipse lie in the directions $\hat{\underline{p}}_j$, $\hat{\underline{p}}_k$ and their length is

(1.42)
$$e_{Tji} = s_{Tji} \left(2F_{2,9;(1-\alpha)}\right)^{\frac{1}{2}} ,$$
$$e_{Tki} = s_{Tki} \left(2F_{2,9;(1-\alpha)}\right)^{\frac{1}{2}} ,$$

where $F_{2,9;(1-\alpha)}$ is $100(1-\alpha)\%$ quantile of the F-distribution on 2 and 9 degrees of freedom.

We shall project the confidence ellipse centrally on the unit sphere and call the thus formed region on the sphere confidence region of the principal direction $\underline{p}_i$.

The angles giving the semiaxes of the confidence region will be denoted e_{ji}, e_{ki},

(1.43)
$$e_{ji} = \tan^{-1} e_{Tji}$$
$$e_{ki} = \tan^{-1} e_{Tki}$$

and called confidence angles of the principal direction $\underline{p}_i$.

The quantities σ_{Tji}, ε_{Tji}, s_{Tji}, s_{ji}, e_{Tji}, e_{ji} $(j, i = 1, 2, 3;\ j \neq i)$, that have been introduced in this and in the preceding sections, are symmetrical in the indeces j, i. We shall accept the convention that the indeces will always be presented in an ascending sequence, e.g. e_{12}, e_{23}, e_{13}.

1.6 ANISOTROPY FACTORS

In many considerations on magnetic susceptibility of rocks we are interested only in the character of the anisotropy, i.e., substantially, in the ratios of the principal susceptibilities, without considering, in the given context, the orientation of the principal directions.

The character of the anisotropy is expressed by means of the so called anisotropy factors; they are non-dimensional quantities derived from the principal susceptibilities [1]).

The following three anisotropy factors can be considered as basic,

$$(1.44) \qquad H_1 = \varkappa_1/\varkappa_2, \quad H_2 = \varkappa_2/\varkappa_3, \quad H_3 = \varkappa_1/\varkappa_3 .$$

Each of the factors H_1, H_2, H_3 is determined by two principal susceptibilities only, any two of the factors are sufficient for a complete description of the character of the anisotropy.

---

[1]) The anisotropy factors can be considered as an analogy or as a special case of a certain class of factors that are commonly used in the literature on geology to characterize quantitatively the fabric of the rock.

Besides of the basic factors mentioned, a series of further factors are known from literature, that are given by more complicated expressions. We shall here consider only three of them that are most necessary for the contemporary interpretation practice,

(1.45)
$$\hat{H}_4 = (\varkappa_1 + \varkappa_2)/(2\varkappa_3)$$
$$\hat{H}_5 = 2\varkappa_1/(\varkappa_2 + \varkappa_3)$$
$$\hat{H}_6 = \varkappa_2^2/(\varkappa_1 \varkappa_3) \quad .$$

The estimates $\hat{H}_i$ ($i = 1, 2, \ldots, 6$) of the anisotropy factors may be obtained when replacing the principal susceptibilities $\varkappa_j$ in the equations (1.44, 1.45) by their estimates $\hat{\varkappa}_j$. Let us suppose that for the estimate $\hat{H}_i$ it holds

(1.46)
$$\hat{H}_i = H_i + dH_i \quad .$$

The variations dH_i will be expressed, in the first approximation, as linear functions of the quantities $d\varkappa_i$, so that

(1.47)

$$\left[dH_1 \;\; dH_2 \;\; dH_3 \;\; dH_4 \;\; dH_5 \;\; dH_6\right]' = \underline{\underline{C}} \left[d\varkappa_1 \;\; d\varkappa_2 \;\; d\varkappa_3\right]' .$$

The matrix $\underline{\underline{C}}$, as can be derived from the equations (1.44, 1.45) has the form

(1.48)

$$\underline{\underline{C}} = \begin{bmatrix} 1/\varkappa_2 & -\varkappa_1/\varkappa_2^2 & 0 \\ 0 & 1/\varkappa_3 & -\varkappa_2/\varkappa_3^2 \\ 1/\varkappa_3 & 0 & -\varkappa_1/\varkappa_3^2 \\ 1/(2\varkappa_3) & 1/(2\varkappa_3) & -(\varkappa_1+\varkappa_2)/(2\varkappa_3^2) \\ 2/(\varkappa_2+\varkappa_3) & -2\varkappa_1/(\varkappa_2+\varkappa_3)^2 & -2\varkappa_1/(\varkappa_2+\varkappa_3)^2 \\ -\varkappa_2^2/(\varkappa_1^2\varkappa_3) & 2\varkappa_2/(\varkappa_1\varkappa_3) & -\varkappa_2^2/(\varkappa_1\varkappa_3^2) \end{bmatrix} .$$

From the linear relation (1.47) there follows for the covariance matrix $^2\underline{\underline{U}}$ of the quantities dH_i

(1.49) $$\sigma^2\underline{\underline{U}} = \sigma^2\underline{\underline{C}}\ \underline{\underline{V}}_1\ \underline{\underline{C}}' ,$$

where $\underline{\underline{V}}_1$ is the covariance matrix of the quantities $d\varkappa_i$, i.e. the upper diagonal (3 x 3) submatrix of the matrix $\underline{\underline{V}}$ given by the relation (1.20).

Dispersions $\sigma^2(H_i)$ of the estimates $\hat{H}_i$ of the anisotropy factors are equal to the diagonal elements of the covariance matrix $\sigma^2\underline{\underline{U}}$,

(1.50)

$$\sigma^2(\hat{H}_1) = 0.4\,\sigma^2(\varkappa_1^2 + \varkappa_2^2 + \tfrac{1}{2}\varkappa_1\varkappa_2)/\varkappa_2^4$$

$$\sigma^2(\hat{H}_2) = 0.4\,\sigma^2(\varkappa_2^2 + \varkappa_3^2 + \tfrac{1}{2}\varkappa_2\varkappa_3)/\varkappa_3^4$$

$$\sigma^2(\hat{H}_3) = 0.4\,\sigma^2(\varkappa_1^2 + \varkappa_3^2 + \tfrac{1}{2}\varkappa_1\varkappa_3)/\varkappa_3^4$$

$$\sigma^2(\hat{H}_4) = 0.1\,\sigma^2\left[(\varkappa_1 + \varkappa_2)^2 + \varkappa_3(\varkappa_1 + \varkappa_2) + \tfrac{1}{2}\varkappa_3^2\right]/\varkappa_3^4$$

$$\sigma^2(\hat{H}_5) = 0.4\,\sigma^2\left[6\varkappa_1^2 + 4\varkappa_1(\varkappa_2 + \varkappa_3) + 4(\varkappa_2 + \varkappa_3)^2\right] / \\ /(\varkappa_2 + \varkappa_3)^4$$

$$\sigma^2(\hat{H}_6) = 0.4\,\sigma^2\left[\varkappa_2^2/(\varkappa_1^4\varkappa_3^4)\right]\left[\varkappa_1^2\varkappa_2^2 + 4\varkappa_1^2\varkappa_3^2 + \varkappa_2^2\varkappa_3^2 + \right. \\ \left. + \varkappa_1\varkappa_2\varkappa_3(\varkappa_1 - \tfrac{1}{2}\varkappa_2 + \varkappa_3)\right] .$$

Estimates $s(\hat{H}_i)$ of the standard errors $\sigma(\hat{H}_i)$ of the anisotropy factors may be obtained from (1.50) by replacing the quantities $\varkappa_i$ and σ by their estimates $\hat{\varkappa}_i$ and s, respectively. The relations thus obtained are rather complicated. We shall, therefore, start from the fact that, in practical computation, the estimates of principal susceptibilities are normalized, i.e. that $(\hat{\varkappa}_1 + \hat{\varkappa}_2 + \hat{\varkappa}_3)/3 = 1$, and that their values differ from one only very little. Under these assumptions simplified approximate relations can be derived.

$$(1.51)\qquad \begin{aligned}
s(\hat{H}_1) &= s(3 + \hat{\varkappa}_1 - 3\,\hat{\varkappa}_2)^{\frac{1}{2}}\\
s(\hat{H}_2) &= s(3 + \hat{\varkappa}_2 - 3\,\hat{\varkappa}_3)^{\frac{1}{2}}\\
s(\hat{H}_3) &= s(3 + \hat{\varkappa}_1 - 3\,\hat{\varkappa}_3)^{\frac{1}{2}}\\
s(\hat{H}_4) &= s\left[0.5(\hat{\varkappa}_1 + \hat{\varkappa}_2 - 5\,\hat{\varkappa}_3 + 4.5)\right]^{\frac{1}{2}}\\
s(\hat{H}_5) &= s\left[0.5(\hat{\varkappa}_1 - 2\,\hat{\varkappa}_2 - 2\,\hat{\varkappa}_3 + 4.5)\right]^{\frac{1}{2}}\\
s(\hat{H}_6) &= s\left[9 + 8\,\hat{\varkappa}_2 - 7(\hat{\varkappa}_1 + \hat{\varkappa}_3)\right]^{\frac{1}{2}} .
\end{aligned}$$

Let us recall that the considerations in this section are bound to the fulfilment of the basic assumptions stated in the section 1.4. If these assumptions are not fulfilled to a sufficient extent, then the estimates of the anisotropy factors are not unbiased. If, for instance, the inequality $\varkappa_1 - \varkappa_2 \gg \sigma$ is not satisfied in a sufficient measure it will cause, in addition to other effects, that the estimate $\hat{H}_1$ will show a systematic bias towards higher values, cf. section 2.2.

1.7 TEST FOR ANISOTROPY

The test for anisotropy serves to verify whether the differences between the principal susceptibilities determined by measurement compared to measuring errors are great enough for us to be entitled to consider the specimen as anisotropic. For anisotropy testing the F-test is used that is in [7] derived from the least square method on the basis of analysis of variance [6].

We have succeeded to construct the test in question in a more simple and a more objective way, that will be now quoted here.

We shall consider as the null hypothesis H_0 the argument that the specimen is isotropic. The argument about the specimen being anisotropic, whether rotationally or triaxially, will represent the alternative hypothesis H_1.

Estimate s^2 of the dispersion σ^2 is independent of the estimates $\hat{k}_i$ of the components of the susceptibility tensor (i = 1, 2, ..., 6). The hypothesis H_0 being valid, it is possible to obtain a new estimate s_1^2 of the dispersion σ^2, that is independent of s^2.

For this purpose we shall introduce auxiliary variables by means of the relation

$$\hat{l}_i = \hat{k}_i + \beta \quad (i = 1, 2, 3) , \tag{1.52}$$

where β is an auxiliary random variable with the distribution N(0, 0.1), independent of $\hat{k}_i$ (i = 1, 2, ..., 6) and s^2. Further, let

$$\hat{\varkappa} = (\hat{k}_1 + \hat{k}_2 + \hat{k}_3)/3 \tag{1.53}$$

designate, as until now, the mean susceptibility, and let

$$\hat{l} = (\hat{l}_1 + \hat{l}_2 + \hat{l}_3)/3 . \tag{1.54}$$

From the covariance matrix $\sigma^2 \underline{\underline{V}}$ (1.20) it follows (without regard to the validity of the hypothesis H_0) that the quantities $\hat{l}_1$, $\hat{l}_2$, $\hat{l}_3$, $\hat{k}_4$, $\hat{k}_5$, $\hat{k}_6$ are mutually independent, the dispersion of the first three being $0.5\sigma^2$ and the dispersion of the remaining being $0.25\sigma^2$.

The zero hypothesis H_o may also be formulated like this

(1.55)
$$E(\hat{i}_1) = E(\hat{i}_2) = E(\hat{i}_3) = E(\hat{i})$$
$$E(\hat{k}_4) = E(\hat{k}_5) = E(\hat{k}_6) = 0 .$$

If H_o holds, then the random variables

(1.56)
$$A = (1/0.5\sigma^2)\sum_{i=1}^{3}(\hat{i}_i - \hat{i})^2 =$$
$$= (1/0.5\sigma^2)\ (k_1^2 + k_2^2 + k_3^2 - 3\varkappa^2) ,$$
$$B = (1/0.25\sigma^2)\ (k_4^2 + k_5^2 + k_6^2)$$

have chi-square distribution on 2 or 3 degrees of freedom, respectively and are mutually independent. The sought estimate s_1^2 of the dispersion σ^2 is thus given by the relation

(1.57)
$$s_1^2 = (A + B)\,\sigma^2/5 .$$

Expression A + B evidently has chi-square distribution on 5 degrees of freedom.

The F-statistic for anisotropy testing

(1.58)

$$F = (s_1^2/s^2) = (2/5s^2)\ (\hat{k}_1^2 + \hat{k}_2^2 + \hat{k}_3^2 - 3\hat{\varkappa}^2 + 2\hat{k}_4^2 + 2\hat{k}_5^2 + 2\hat{k}_6^2)$$

then has F-distribution on 5 and 9 degrees of freedom. It is evident that in the case of validity of the H_1 hypothesis, the estimate s_1^2 as well as the F-statistic have a tendency to take on values higher than in the case of validity of the hypothesis H_o. We shall reject the hypothesis H_0 (about the specimen being isotropic) in favour of the alternative hypothesis H_1 (about the specimen being anisotropic) on the significance level α, if

$$(1.59) \qquad F > F_{5,9;\,(1-\alpha)} \,,$$

where the expression on the right side of the inequality is $(1 - \alpha)\%$ quantile of the F-distribution on 5 and 9 degrees of freedom.

In the table 1.1 values of these quantiles for several levels of significance are given [5].

Table 1.1

α [%]	$1 - \alpha$ [%]	$F_{5,9;(1-\alpha)}$
10	90	2.6106
5	95	3.4817
2.5	97.5	4.4844
1	99	6.0569

The relation (1.58) for F-statistic is invariant with respect to the orthonormal transformation of coordinates, which is the consequence of the choice of the rotatable measuring design. This property can also be proved by means of invariants of quadratic forms, see e. g. [1]. There is some advantage in expressing F-statistic in a coordinate system determined by the vectors of principal directions. Thus we get a simpler relation

$$(1.60) \qquad F = (2/5s^2)\,(\hat{\varkappa}_1^2 + \hat{\varkappa}_2^2 + \hat{\varkappa}_3^2 - 3\hat{\varkappa}^2) \,,$$

which may be arranged into the form

$$(1.61) \qquad F = (2/5s^2)\sum_{i=1}^{3}(\hat{\varkappa}_i - \hat{\varkappa})^2 .$$

Let us notice that the construction of the described F-test is quite exact, no simplifying assumption being used.

There is another meaning to the F-statistic. A sufficiently high value of the F-statistic is a necessary condition for the simplifying assumption formulated at the beginning of the section 1.4 to be fulfilled. It is necessary to emphasize that the high value of the F-statistic is not a sufficient condition as it does not exclude the case of rotational anisotropy, that will be discussed in greater detail in the next chapter.

2. TWO-DIMENSIONAL PROBLEM

2.1 TRANSITION TO TWO-DIMENSIONAL PROBLEM

In practical applications we often come across a case of an approximately rotational anisotropy, where either the principal susceptibility $\varkappa_3$ or $\varkappa_1$ significantly differ from the two remaining principal susceptibilities, i.e. there holds either

$$(2.1) \qquad \varkappa_2 - \varkappa_3 \gg \varkappa_1 - \varkappa_2 ,$$

or

$$(2.2) \qquad \varkappa_1 - \varkappa_2 \gg \varkappa_2 - \varkappa_3 .$$

This type of anisotropy is usual in quite a series of rock types. In many sediments, for example, the principal susceptibility correspon- ding to the direction perpendicular to S-planes differ significantly from the other principal susceptibilities which almost coincide.

For the sake of simplicity we shall limit ourselves only to the case fulfilling (2.1), as the second case may be discussed quite ana- logically.

We shall not, for the sake of a clear record, in the quantities introduced in this chapter denote that they concerne the case fulfil- ling (2.1). There will be, however, formal reasons for making an ex- ception in the denotation of the confidence angles of the principal directions and of the statistics for anisotropy testing.

In the meantime, let us suppose that the conditions for the va- lidity for Hext's statistics are fulfilled, i.e. that in the chosen case

$$(2.3) \qquad \varkappa_1 - \varkappa_2 \gg \sigma .$$

According to the considerations in section 1.4 we shall come to the conclusion that the tolerance regions of the principal directions $\hat{\underline{p}}_1$, $\hat{\underline{p}}_2$ are very elongated, that their longer axes are lying in the plane ($\underline{p}_1$, $\underline{p}_2$) and, further, that the tolerance region of the $\hat{\underline{p}}_3$ direc- tion is substancially smaller its form being approximately circular. With regard to the relatively small variations of the estimated prin- cipal direction $\hat{\underline{p}}_3$ there is some sense in simplifying the considera- tion by neglecting the variations of the direction $\hat{\underline{p}}_3$. This narrows the three-dimensional problem of the statistic estimation of the sus- ceptibility tensor to a two-dimensional problem in the ($\underline{p}_1$ $\underline{p}_2$) plane; within the limits of the simplification used, this plane coincides with the ($\hat{\underline{p}}_1$ $\hat{\underline{p}}_2$) plane.

Let us now consider what happens when the assumption (2.1) is fulfilled, but the assumption (2.3) is not fulfilled to a sufficient extent. It is evident from observation, and it is relatively simple to prove, that, even in this case, the direction $\hat{p}_3$ will show small variations. Further, it may be expected that the tolerance regions of the $\hat{p}_1$, $\hat{p}_2$ directions will be still more elongated. At the same time it is clear, that for a limit case of a complete rotational anisotropy (for $\varkappa_1 = \varkappa_2$) the tolerance regions of the $\hat{p}_1$, $\hat{p}_2$ directions will coincide and form on the unit sphere a narrow girdle around the great circle determined by the plane ($\hat{p}_1$ $\hat{p}_2$). The variations of the directions $\hat{p}_1$, $\hat{p}_2$ will thus be still greater than in the case when the condition (2.3) is fulfilled; it is all the more possible to neglect the variations of the direction $\hat{p}_3$ and pass to the two-dimensional task as shown above.

The aim of this chapter is to discuss the two-dimensional problem mentioned, and to do so - with a small exception - without the limiting condition (2.3). The considerations of this chapter surpass the scope of the problems discussed in [7] and represent an extension of the theory of anisotropy measurement into a region that is of considerable importance for the practice.

We shall now try to describe the transition to the two-dimensional problem from the point of view of the tensor notation. We shall just start from the knowledge that a tensor may be "perpendicularly projected" into a plane, similarly to a vector.

Let $\hat{k}$ denote the susceptibility tensor estimated from the rotatable design of 15 measuring directions. Let its matrix expression in the system $\{P\}$ determined by the principal directions p_1, p_2, p_3 be

(2.4) $$\hat{\underline{\underline{k}}}^p = \begin{bmatrix} \varkappa_1+dk_1 & dk_4 & dk_6 \\ dk_4 & \varkappa_2+dk_2 & dk_5 \\ dk_6 & dk_5 & \varkappa_3+dk_3 \end{bmatrix} .$$

We shall project the tensor $\hat{\underline{\underline{k}}}$ into the plane ($\underline{p_1}$ $\underline{p_2}$). In this way we shall obtain the two-dimensional tensor $\hat{\underline{\underline{K}}}$ describing the properties of the susceptibility in the plane mentioned. Tensor $\hat{\underline{\underline{K}}}$ will be, in the coordinate system determined by the directions $\underline{p_1}$ and $\underline{p_2}$ expressed by a square matrix [1])

(2.5) $$\hat{\underline{\underline{K}}} = \begin{bmatrix} \varkappa_1+dk_1 & dk_4 \\ dk_4 & \varkappa_2+dk_2 \end{bmatrix}$$

or, alternatively, by a column matrix

(2.6) $$\hat{\underline{K}} = \begin{bmatrix} \varkappa_1+dk_1 & \varkappa_2+dk_2 & dk_4 \end{bmatrix}' .$$

The introduction of the tensor $\hat{\underline{\underline{K}}}$ denotes the transition to a two--dimensional case in the above mentioned sense. The plane ($\underline{p_1}$ $\underline{p_2}$), in which the tensor is lying, is namely identified with the plane ($\hat{\underline{p}}_1$ $\hat{\underline{p}}_2$), while the variations of the direction $\hat{\underline{p}}_3$ are thus neglected.

We shall express the variations of tensor $\hat{\underline{\underline{K}}}$ by the matrix

(2.7) $$d\underline{K} = \begin{bmatrix} dk_1 & dk_2 & dk_4 \end{bmatrix}' .$$

----

[1]) The system determined by the directions $\underline{p_1}$, $\underline{p_2}$ is here considered as fundamental. That is why the matrices expressing the susceptibility tensor are not indexed in this system.

The elements of this matrix have central normal distribution with the covariance matrix

(2.8)
$$\sigma^2 \underline{\underline{V}}_2 = \sigma^2 \begin{bmatrix} 0.4 & -0.1 & \\ -0.1 & 0.4 & \\ & & 0.25 \end{bmatrix},$$

as follows from the normal distribution of the elements of the matrix $\hat{\underline{\underline{k}}}^p$ and from the covariance matrix $\sigma^2 \underline{\underline{V}}$ (1.20).

The estimated principal susceptibilities $\hat{\varkappa}_1$, $\hat{\varkappa}_2$ and the principal directions $\hat{\underline{p}}_1$, $\hat{\underline{p}}_2$ satisfy the equation

(2.9)
$$\hat{\underline{\underline{k}}}\, \hat{\underline{p}}_i = \hat{\varkappa}_i\, \hat{\underline{p}}_i ,$$

the estimated principal direction $\hat{\underline{p}}_i$ being expressed by the column matrix

(2.10)
$$\hat{\underline{p}}_i = \begin{bmatrix} \hat{p}_{1i} & \hat{p}_{2i} \end{bmatrix}' .$$

2.2 VARIATIONS OF PRINCIPAL SUSCEPTIBILITIES

Let us seek the estimate $\hat{\varkappa}_1$, $\hat{\varkappa}_2$ of the principal susceptibilities in the two-dimensional case. By solving the relation (2.9) we get for the susceptibilities $\hat{\varkappa}_1$, $\hat{\varkappa}_2$ the secular equation

(2.11)
$$\hat{\varkappa} 2 - \hat{\varkappa}(\varkappa_1 + dk_1 + \varkappa_2 + dk_2) + (\varkappa_1 + dk_1)(\varkappa_2 + dk_2) - dk_4^2 = 0 ,$$

the roots of which are

(2.12) $$\hat{\varkappa}_{1,2} = \tfrac{1}{2}(\varkappa_1 + \varkappa_2 + dk_1 + dk_2 \pm \hat{r}) ,$$

where

(2.13) $$\hat{r} = \hat{\varkappa}_1 - \hat{\varkappa}_2 = \left[(r + \sigma v)^2 + (\sigma w)^2\right]^{\frac{1}{2}} ,$$

where

$$\hat{r} = \varkappa_1 - \varkappa_2, \quad v = (dk_1 - dk_2)/\sigma , \quad w = 2\, dk_4/\sigma .$$

Random variables v and w are independent and their distribution is N(0, 1), that immediatly follows from the covariance matrix $\sigma^2 \underline{\underline{V}}_2$, given in(2.8).

Thus the difference of the estimated principal susceptibilities $\hat{r}$ has the same distribution as the length of the radius vector of a point in a plane, the cartesian coordinates $(r + \sigma v)$, (σw) of which are independent, their distribution being $N(r, \sigma^2)$ and $N(0, \sigma^2)$, respectively. So the distribution of the random variable $\hat{r}$ is the non-central Rayleigh distribution with the excentricity parameter r. The random variable

(2.14) $$\hat{\rho} = \hat{r}/\sigma = \left[(\rho + v)^2 + w^2\right]^{\frac{1}{2}}$$

then has normalized non-central Rayleigh distribution with the excentricity parameter $\rho = r/\sigma$ and with the probability density

(2.15) $$f(\hat{\rho}) = \hat{\rho} \exp\left[-(\hat{\rho}^2 + \rho^2)/2\right] I_0(\hat{\rho}\,\rho) ,$$

where $I_0(.)$ denotes the Bessel function of the zero order and imaginary argument, see [3].

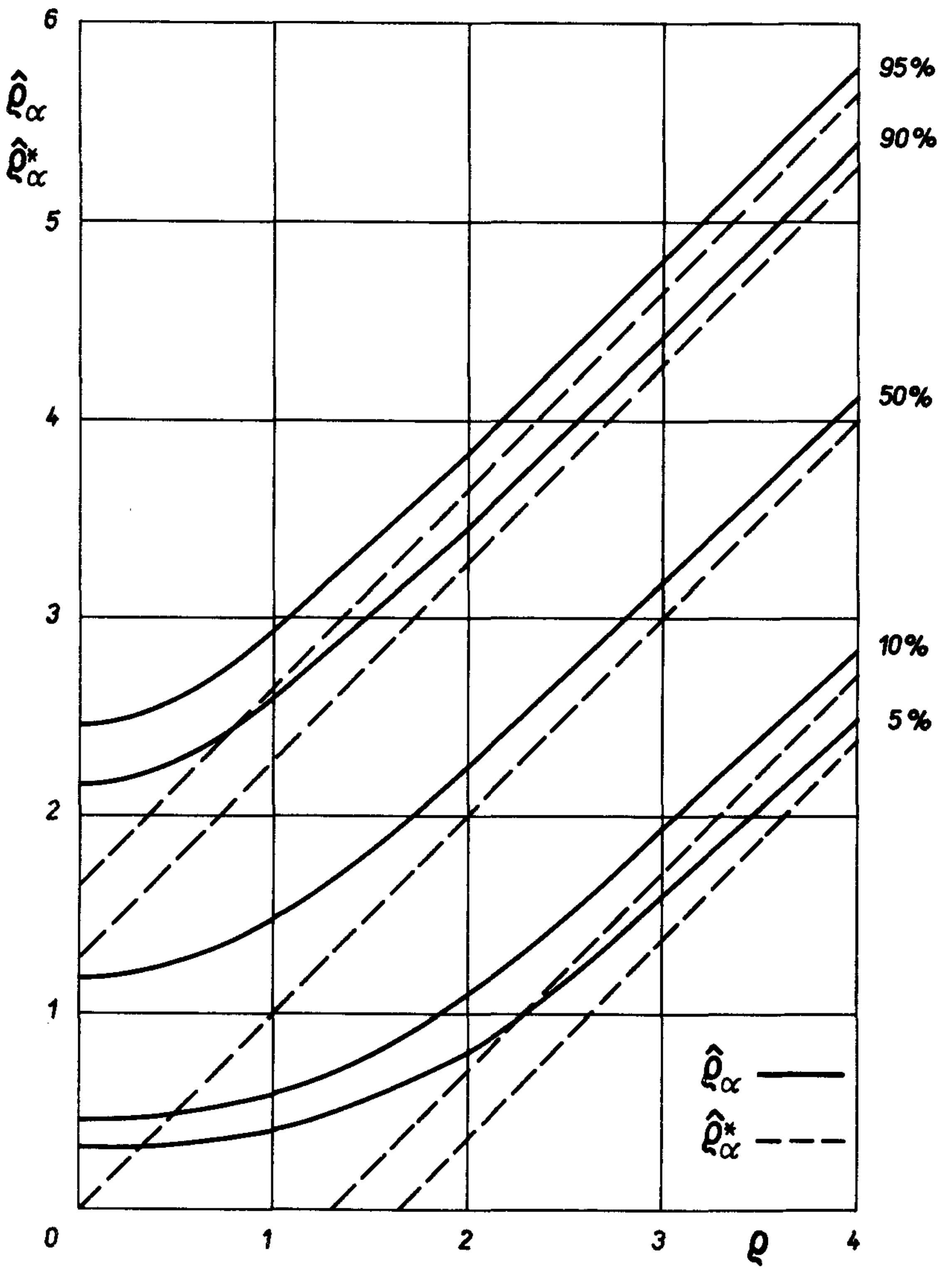

Fig. 2. 1 Dependence of 100α % quantiles of random variable $\hat{\rho}$ with normalized non--central Rayleigh distribution on the excentricity parameter ρ ; dependence of the quantiles of random variable $\hat{\rho}^*$ with N (ρ , 1) distribution on the parameter ρ .

In figure 2.1 the dependence of 100α% quantiles of the random variable $\hat{\rho}$ on the excentricity parameter ρ (α = 5, 10, 50, 90 and 95 %) is shown [1]. For the sake of better illustration also the analogous dependence of the quantiles of the random variable ρ^{*} with the distribution N(ρ,1) on ρ is shown.

From the figure it can be seen that the difference of the estimated principal susceptibilities $\hat{r}$ is systematically greater than that of the real principal susceptibilities r. Thus the measuring errors apparently increase the degree of anisotropy. Nevertheless, this effect shows only if the anisotropy is small with regard to the standard error of measurement when the condition (2.3) is not fulfilled to a sufficient extent, i.e. for small values ρ, say for $\rho < 3$.

2.3 CONFIDENCE INTERVAL OF THE DIFFERENCE OF PRINCIPAL SUSCEPTIBILITIES

Fig. 2.1. may serve for the construction of confidence intervals for ρ and r on the basis of estimates $\hat{\rho}$ or $\hat{r}$. As for the standard error measurement σ its estimate s is substituted, the obtained confidence intervals are of an approximate character only.

For illustration we shall give two examples :

1) By measuring it was found $\hat{\rho} \doteq \hat{r}/s = 2.4$. The approximate 80 % confidence interval for r will be given by the inequality

$$0.7s \leqq r \leqq 3.5s \quad .$$

----

[1]) The curves have been constructed on the basis of the table given in [4].

2) By measuring it was found $\hat{\rho} = \hat{r}/s = 1.4$. In this case it has some sense to construct only a one-sided confidence interval. The approximate one-sided 90% confidence interval will be given by the inequality

$$r \leqq 2.4s .$$

2.4 VARIATIONS OF PRINCIPAL DIRECTIONS

The variations of the estimated principal direction $\hat{\underline{p}}_1$ in the plane ($\underline{p}_1$ $\underline{p}_2$) will be judged according to the angle

$$\omega = \tan^{-1} (\hat{p}_{21}/\hat{p}_{11}) , \tag{2.16}$$

that is formed by this direction and the actual direction $\underline{p}_1$, see fig. 2.2. (It is evident that the angle ω simultaneously gives the magnitude of variations of the estimated principal direction $\hat{\underline{p}}_2$.)

In the matrix equation (2.9) we shall replace the general index i by the index 1. From the matrix equation we shall obtain two ordinary equations from which we shall select the following

$$dk_4\hat{p}_{11} + (\varkappa_2 + dk_2 - \hat{\varkappa}_1)\hat{p}_{21} = 0 . \tag{2.17}$$

For $\hat{\varkappa}_1$ we shall substitute according to (2.12); after arrangements by using (2.13) we get

$$\sigma w\hat{p}_{11} - \left\{r + \sigma v - \left[(r + \sigma v)^2 + (\sigma w)^2\right]^{\frac{1}{2}}\right\} \hat{p}_{21} = 0 . \tag{2.18}$$

From (2.16; 2.18) we shall get for the sought angle

$$\omega = \tan^{-1} \frac{z}{1 + \eta (+ z^2)^{\frac{1}{2}}} \quad , \tag{2.19}$$

where

$$z = \frac{\sigma w}{r + \sigma v} \quad , \tag{2.20}$$

$$\eta = \text{sign } (r + \sigma v) \ . \tag{2.21}$$

The last equation may be also written in the form

$$\eta = \text{sign } (\varkappa_1 + dk_1 - \varkappa_2 - dk_2) \ . \tag{2.21a}$$

The random variables v, w are, as already stated, independent and their distribution is N(0, 1).

Equation (2.19) can further be arranged,

$$\omega = \tfrac{1}{2} \tan^{-1} z \ , \tag{2.22}$$

where for $\eta = 1$ the principal branch of the function $\tan^{-1}$ and for $\eta = -1$ the subsidiary branches illustrated in fig. 2.3 should be taken.

It can be seen that the angle ω may deviate from the interval $\langle -\pi/4, \ \pi/4 \rangle$. This phenomenon occurs when $\eta = -1$, i.e. when $(\varkappa_1 + dk_1) < (\varkappa_2 + dk_2)$. It is interesting that the occurrence of the phenomenon mentioned is decided only by the diagonal elements of the matrix $\hat{\underline{K}}$.

Using the earlier defined error parameter σ_{Tji}, see (1.31), the following expression can be derived for the quantities z, η

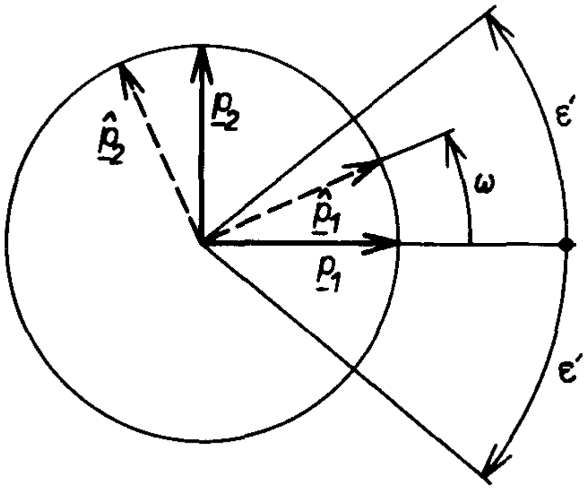

Fig. 2. 2 Variations of the principal directions of susceptibility in the two-dimensional case.

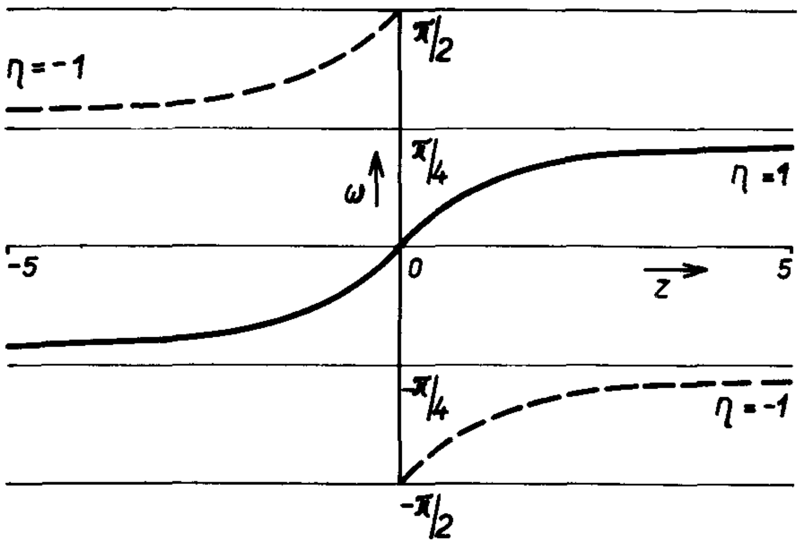

Fig. 2.3 Function $\omega = \frac{1}{2} \tan^{-1} z$.

(2.23)
$$z = \frac{2\sigma_{T12}w}{1 + 2\sigma_{T12}v},$$

(2.24)
$$\eta = \text{sign}\,(1 + 2\sigma_{T12}v)\ .$$

We shall further seek the angle tolerance interval $\langle -\varepsilon', \varepsilon' \rangle$, that will exclude 100α % of the least likely positions of the principal direction $\hat{\underline{p}}_1$. A general construction of this interval would be rather complicated, which is why we are discussing only certain significant cases.

A) We shall first suppose that the error parameter $\sigma_{T12} \ll 1$. Then it approximately holds

(2.25)
$$\omega = \sigma_{T12}w\ ,$$

so that

(2.26)
$$\varepsilon' = \sigma_{T12}\, u_{(1-\alpha/2)}\ ,$$

where $u_{(1-\alpha/2)}$ is $100(1 - \alpha/2)$% quantile of the distribution N(0, 1).

B) Let $\sigma_{T12} \rightarrow \infty$. In this case the random variable ω has uniform distribution on the interval $(-\pi/2, \pi/2)$, so that

(2.27)
$$\varepsilon' = (\pi/2)\,(1 - \alpha/2)\ .$$

C) Finally, let us seek such a value of the error parameter σ_{T12}, for which $\varepsilon' = \pi/4$. According to (2.22) the angle ω will deviate from the interval $\langle -\pi/4,\ \pi/4 \rangle$, if and only if $\eta = -1$. According to (2.24) the probability of the phenomenon $\eta = -1$ is the same as that of the random variable with N(0, 1) distribution exceeding the value $1/2\,\sigma_{T12}$. Hence we shall immediately derive that the angle interval $\langle -\pi/4,\ \pi/4\rangle$ excludes 100 α % of the least likely positions of the principal direction $\hat{\underline{p}}_1$, if the value of the error parameter is

$$\sigma_{T12} = 1/2u_{(1-\alpha)} \ , \tag{2.28}$$

where $u_{(1-\alpha)}$ is 100(1 - α)% quantile of N(0, 1) distribution.

2.5 CONFIDENCE INTERVALS OF THE PRINCIPAL DIRECTION FOR SMALL MEASURING ERRORS

Let us seek the angle confidence interval for the actual principal direction $\underline{p}_1$ in the considered two-dimensional case under the condition that the assumptions of the Hext approximation are fulfilled,which means that the inequality (2.3) is fulfilled.

Under the condition stated the random variable ω , characterizing the variations of the estimated direction $\hat{\underline{p}}_1$, is given by the equation (2.25). Let us divide this equation by the estimate of the error parameter s_{T12} given by (1.38),

$$\frac{\omega}{s_{T12}} = w\,\frac{\sigma/|\varkappa_1 - \varkappa_2|}{s/|\hat{\varkappa}_1 - \hat{\varkappa}_2|} \ . \tag{2.29}$$

Random variable w has distribution N(0, 1) and is independent of the expression s/σ . This expression is distributed as $(\chi^2/9)^{\frac{1}{2}}$, where χ^2 has chi-square distribution on 9 degrees of freedom. Within the approximation considered, the equality $\varkappa_1 - \varkappa_2 = \hat{\varkappa}_1 - \hat{\varkappa}_2$ is fulfilled. Hence it follows that the random variable ω/s_{T12} has Student distribution on 9 degrees of freedom. As a consequence, the 100(1 - α')%

angle confidence interval for the actual direction $\underline{p}_1$ has in angular measure the half width

$$(2.30) \qquad e'_{12} = s_{T12}\, t_{9;(1-\alpha'/2)}$$

and is symmetrically laid round the estimated principal direction $\hat{\underline{p}}_1$. The angle e'_{12} will be called confidence angle for the cosidered two-dimensional case.

The confidence angle e_{12} for the three-dimensional case is, in the approximation considered, given by the equation

$$(2.31) \qquad e_{12} = s_{T12}\left[2F_{2,9;(1-\alpha)}\right]^{\frac{1}{2}},$$

that arises by the linearization of the equation (1.43).

If the level α for the confidence angle e_{12} is given, it is possible, on the basis of the equations (2.30, 2.31), to find the level α' for the confidence angle e'_{12} so as to make $e'_{12} = e_{12}$. For example for α = 10 or 5 %, α' = 4 or 2 %, respectively. This consideration shows a further connection between the two-dimensional and three-dimensional problem.

2.6 TEST FOR ANISOTROPY IN TWO-DIMENSIONAL CASE

The test for anisotropy discussed in section 1.7 allows to decide whether we are entitled, on the basis of the results of measurement, to consider the specimen as anisotropic. It does not, however, permit to distinguish the type of anisotropy, i.e. whether it is the case of the rotational or triaxial anisotropy.

But if the specimen has either a significant minimal susceptibility $\varkappa_3$ or maximal susceptibility $\varkappa_1$, see section 2.1, the type of anisotropy may be judged on the strength of the test for anisotropy in the plane ($\underline{p}_1$, $\underline{p}_2$) or ($\underline{p}_2$, $\underline{p}_3$), respectively. We shall limit ourselves to the first case only; the test dealt with in the second case is analogous.

The null hypothesis H_0 will be the statement the the specimen is rotationally anisotropic around the axis $\underline{p}_3$, i.e. that $\varkappa_1 = \varkappa_2$. The alternative hypothesis H_1 will consist in the statement that the specimen is triaxially anisotropic, i.e. that $\varkappa_1 \neq \varkappa_2$.

If H_0 holds, the random variable $\hat{\rho} = (\hat{\varkappa}_1 - \hat{\varkappa}_2)/\sigma$ has normalized central Rayleigh distribution, see section 2.2, so that $\hat{\rho}^2$ has chi-square distribution on 2 degrees of freedom. The random variable $(s/\sigma)^2$ (without regard to the validity of H_0) is independent of $\hat{\rho}$ and it is distributed as $\chi^2/9$, where the distribution of χ^2 is chi--square on 9 degrees of freedom.

Statistic

$$F_{12} = \tfrac{1}{2}(\hat{\varkappa}_1 - \hat{\varkappa}_2)^2/s^2 \tag{2.32}$$

therefore has, in the case of validity of H_0, F-distribution on 2 and 9 degrees of freedom.

We shall reject the null hypothesis H_0 of the rotational anisotropy in favour of the alternative hypothesis H_1 of the triaxial anisotropy on the significance level α, if

$$F_{12} > F_{2,9;(1-\alpha)} \ , \tag{2.33}$$

where $F_{2,9;(1-\alpha)}$ is $(1-\alpha)$% quantile of the F-distribution on 2 and 9 degrees of freedom.

Table 2.1 gives the values of these quantile for several significance levels [5].

Tab. 2.1

α [%]	$1-\alpha$ [%]	$F_{2,9;(1-\alpha)}$
10	90	3.0065
5	95	4.2565
2,5	97.5	5.7147
1	99	8.0215

There is still another way of performing this test, namely by using confidence angles. From the equations (2.32, 1.38, 1.42, 1.43) it may be derived for the statistic F_{12}

$$(2.34 \qquad F_{12} = \frac{F_{2,9;(1-\alpha)}}{4\tan^2 e_{12;(1-\alpha)}} ,$$

where $e_{12;(1-\alpha)}$ is $(1-\alpha)\%$ confidence angle for the three-dimensional case.

From the equations (2.33, 2.34) results the following criterion, equivalent to the above mentioned criterion : the hypothesis H_0 will be rejected in favour of H_1 on the significance level α if

$$(2.35) \qquad e_{12;(1-\alpha)} > \tan^{-1}\frac{1}{2} \approx 26.5^\circ .$$

This result is too surprisingly simple and very useful. As far as the confidence angle e_{12} for the considered value α is known, it is sufficient to compare it simply with the constant angle 26.5°.

Hext [7] suggests a test for coincidence of two eigenvalues (principal susceptibilities) that should fulfil the same function as the test described in this section. However, the construction of his test is not correct. The error was caused by the fact that the proposed testing statistic has the assumed Student distribution only in the case when the difference of any two eigenvalues (principal susceptibilities) is sufficiently great in comparison with the measuring errors, see the fundamental assumption stated in section 1.4. Thus it is not possible to use such a statistic for testing the coincidence of two eigenvalues.

3. EXPRESSING SUSCEPTIBILITY PARAMETERS IN VARIOUS COORDINATE SYSTEMS

Until now we have expressed susceptibility parameters in the coordinate system of the specimen or, in some cases, in the coordinate system of principal directions. To be able to compare mutually various specimens (e.g. specimens taken from the same geological body) it is necessary to introduce a suitable common coordinate system, which can be either the geographic or the tectonic coordinate system. More detailed characteristics of the coordinate system of the specimen, the geographic and tectonic systems, are given in section 3.2.

To the susceptibility parameters that will necessarily have to be transformed between the systems stated belong the susceptibility tensor and the vectors of principal directions. The principal susceptibilities and the mean susceptibility are scalar quantities and thus they are not bound to any system.

The transformations equations in this chapter are written for the actual (correct) parameter values. In a practical situation, however, we substitute estimated (measured) values.

This indicates that the strict distinguishing between the actual and estimated values, maintained until now, becomes redundant in this phase. This is the reason why - as far as nothing prevents it - the estimated values will be treated as if they were actual values. This will also show in nomenclature, as well as in symbolics. Thus, e.g., we shall only write "susceptibility tensor $\underline{\underline{k}}$" instead of "estimated susceptibility tensor $\hat{\underline{\underline{k}}}$, and others like that. Such a simplified way of expressing is, after all, quite usual in applications of the theory of measurement.

The above note concerns, to a reasonable extent, also the following chapter 4.

3.1 ORTHONORMAL TRANSFORMATION OF VECTOR AND TENSOR

In this section we shall draw the attention to the fundamental equations for the transformation of components of the vector and tensor between two cartesian coordinate systems. Notes on the symbolics used were given in the introduction.

Let us consider a coordinate system $\{Y\}$ with the axes y_1, y_2, y_3 and a system $\{Z\}$ with the axes z_1, z_2, z_3. The transformation matrix of the system $\{Y\}$ to the system $\{Z\}$ will be denoted $\underline{\underline{T}}^{ZY}$. It is determined by the equation

$$(3.1) \qquad \underline{\underline{T}}^{ZY} = \begin{bmatrix} \cos(z_1, y_1) & \cos(z_1, y_2) & \cos(z_1, y_3) \\ \cos(z_2, y_1) & \cos(z_2, y_2) & \cos(z_2, y_3) \\ \cos(z_3, y_1) & \cos(z_3, y_2) & \cos(z_3, y_3) \end{bmatrix} .$$

Let $\underline{h}$ be a certain vector, that is in $\{Y\}$ and in $\{Z\}$ expressed by a column matrix $\underline{h}^Y$ or $\underline{h}^Z$, respectively. Then it holds,

$$(3.2) \qquad \underline{h}^Z = \underline{\underline{T}}^{ZY} \underline{h}^Y \quad .$$

Let $\underline{\underline{k}}$ be a certain tensor of the 2nd order that is in $\{Y\}$ and $\{Z\}$ expressed by a square matrix $\underline{\underline{k}}^Y$ or $\underline{\underline{k}}^Z$, respectively. Then it holds

(3.3) $$\underline{\underline{k}}^{Z} = \underline{\underline{T}}^{ZY}\, \underline{\underline{k}}^{Y}\, (\underline{\underline{T}}^{ZY})' \, .$$

The transformation matrix $\underline{\underline{T}}^{ZY}$ of the system $\{Y\}$ to the system $\{Z\}$ is orthonormal under the chosen assumptions. For the matrix $\underline{\underline{T}}^{YZ} = (T^{ZY})^{-1}$ that is the transformation matrix of $\{\bar{Z}\}$ to $\{\bar{Y}\}$, it holds

(3.4) $$\underline{\underline{T}}^{YZ} = (\underline{\underline{T}}^{ZY})' \, .$$

3.2 CHARACTERISTIC OF THE USED COORDINATE SYSTEMS

The coordinate system of the specimen is a system in which the anisotropy measurement, as well as the fundamental numerical processing are carried out. It is the system that is firmly bound to the specimen and usually it is marked on its surface in a suitable way. The system of the specimen will be denoted $\{X\}$, its axes x_1, x_2, x_3. Though, in principle, the system of the specimen may be chosen in an arbitrary way, a certain usage is maintained. In cubic specimens that are generally used for anisotropy measurement, the system of the specimen is chosen in such a way that its axes coincide with the edge of the specimen, see fig. 1.2. A further principle will be given below, in connection with the definition of the angles of sampling.

The coordinate system of the specimen is considered as "fundamental" in the sense of the note on the symbolics in the introductiom; the matrices expressing vectors and tensors are not indexed in this system.

The geographic system of coordinates will be denoted Y ,

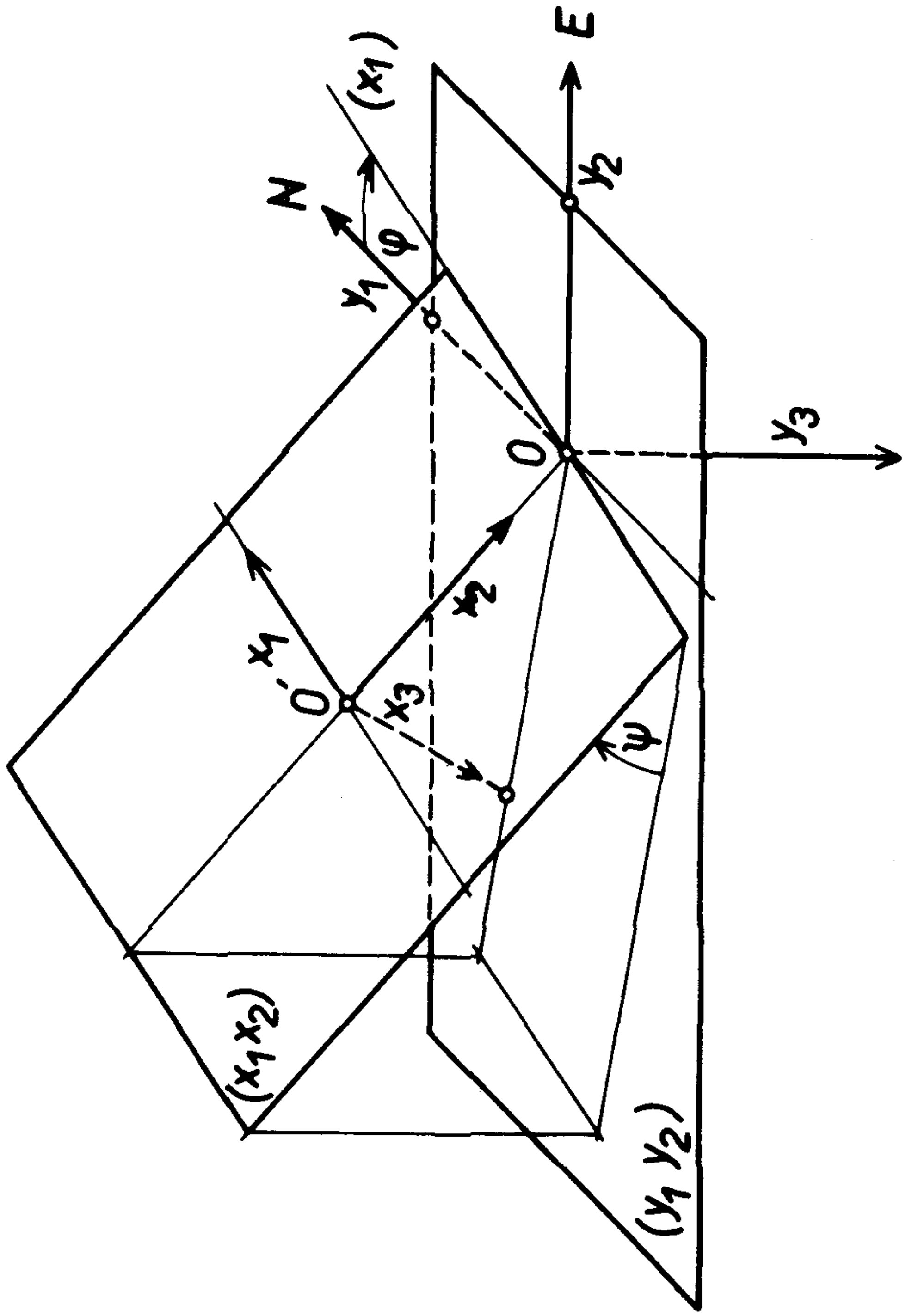

Fig. 3. 1 Orientation of sampling, i.e. orientation of the coordinate system of the specimen {X} with respect to the geographic coordinate system {Y}, expressed by the sampling angles φ, ψ.

its axes y_1, y_2, y_3. Axes y_1, y_2 lie in the horizontal plane, the first of them aiming to the north, the second to the east. The axis y_3 aims vertically downwards.

The anisotropy of magnetic susceptibility is measured on oriented specimens. That means that the orientation of the coordinate system of the specimen $\{X\}$ with regard to the geographical system $\{Y\}$ is known. The orientation is expressed by the so called sampling angles φ and γ, as illustrated in fig. 3.1. At the same time $\varphi \in \langle 0^\circ, 360^\circ)$, $\gamma \in \langle 0^\circ, 180^\circ\rangle$. The coordinate system of specimen is chosen in such a way that in the original position of the specimen (in situ) the x_1 axis is horizontal. That is why two angles are sufficient for the determination of orientation. Concrete sampling procedure of specimen can be rather varied, but their description is not subject of this work.

The tectonic system expresses, roughly speaking, three tectonically significant, mutually perpendicular direction in the rock at the place of sampling. In geology it is customary to denote the axes of the tectonic system as a, b, c. The (a b) plane is usually the foliation plane, the b axis lies in the lineation direction, the c axis is perpendicular to the (a b) plane. (This characteristic of the tectonic system is exclusively informative, for a more exact explanation should be consulted in professional literature, see e.g. [16].)

Not wanting to break the homogeneity of the symbolics, we shall denote the tectonic system as $\{Z\}$, its axes as z_i, where $z_1 \equiv a$, $z_2 \equiv b$, $z_3 \equiv c$. As the tectonic system is bound to the rock in the place of sampling, it is also bound to the specimen taken. In spite of that, its orientation is usually considered with regard to the geographic system, not to the system of the specimen, and expressed by a triad of angles λ, μ, ν as illustrated in fig. 3.2. At the same time $\lambda \in \langle 0^\circ, 360^\circ)$, $\mu \in \langle 0^\circ, 180^\circ\rangle$, $\nu \in \langle 0^\circ, 180^\circ)$.

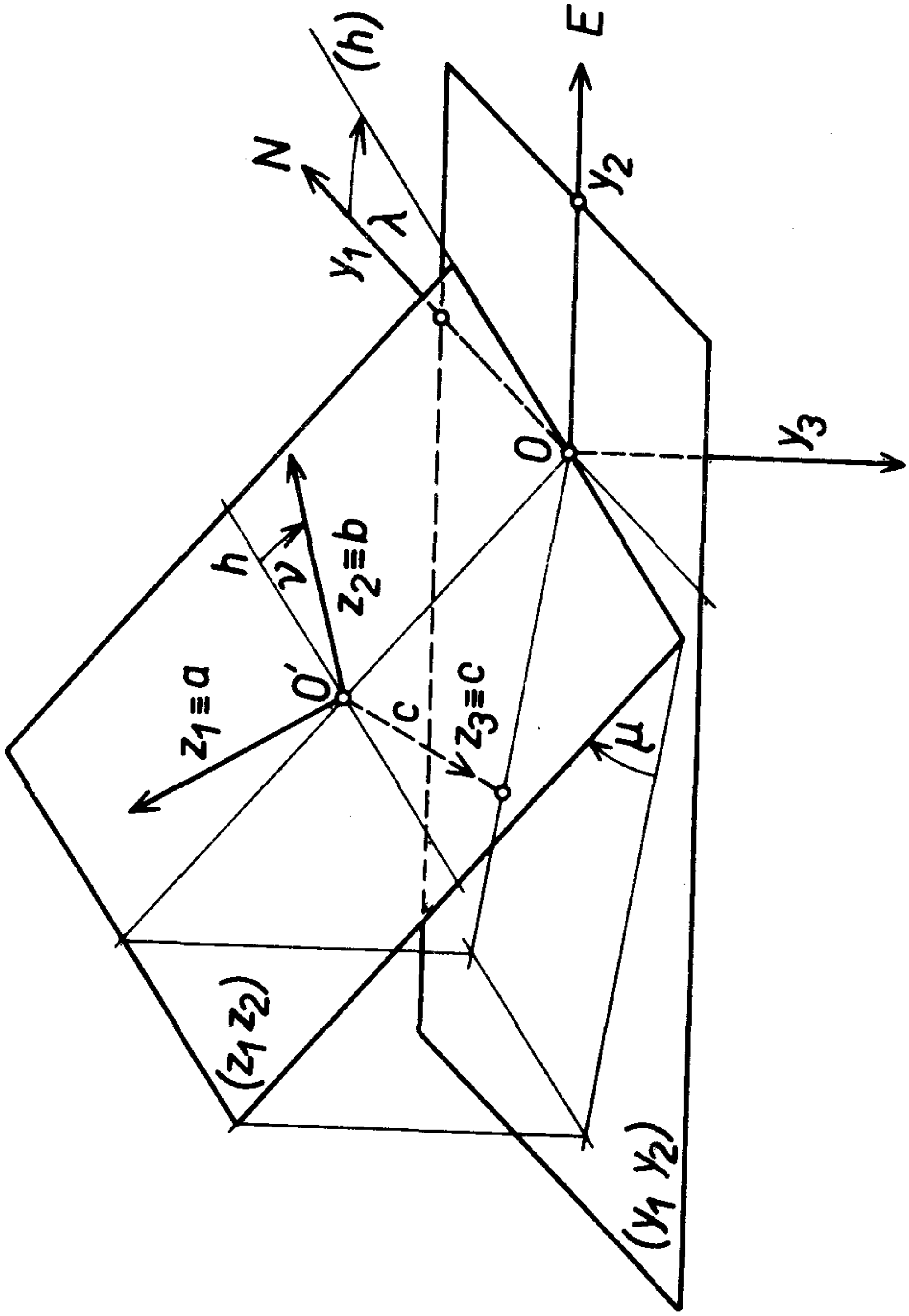

Fig. 3. 2 Orientation of the tectonic coordinate system {Z} with respect to the geographic coordinate system {Y}, expressed by the angles λ, μ, ν.

In transformations we shall also consider the coordinate system determined by the principal directions $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$, and denote it $\{P\}$. Whereas the coordinate systems of the specimen, the geographic and the tectonic one, are altogether righthanded, the coordinate system determined by principal directions need not be righthanded.

3.3 TRANSFORMATION RELATIONS FOR SUSCEPTIBILITY PARAMETERS

A survey of relations between susceptibility parameters expressed in various coordinate systems can be found in fig. 3.3.

The matrix $\underline{\underline{k}}^P$ expressing the susceptibility tensor in the system of principal directions has the following simple form

$$\underline{\underline{k}}^P = \begin{bmatrix} \varkappa_1 & 0 & 0 \\ 0 & \varkappa_2 & 0 \\ 0 & 0 & \varkappa_3 \end{bmatrix} . \tag{3.5}$$

The matrix $\underline{\underline{p}}$ contains in its columns directional cosines of the principal directions in the system of specimen,

$$\underline{\underline{p}} = \begin{bmatrix} \underline{p}_1 & \underline{p}_2 & \underline{p}_3 \end{bmatrix} ,$$

see also (1.6, 1.7). This matrix can obviously be understood as a transformation matrix of the system of principal directions $\{P\}$ to the system of specimen $\{X\}$. The form and meaning of the matrices $\underline{\underline{p}}^Y$ and $\underline{\underline{p}}^Z$ is analogous.

The matrix $\underline{\underline{T}}^{YX}$ is the transformation matrix of the system of the specimen $\{X\}$ to the geographic system $\{Y\}$. On the basis of the earlier stated definition of the sampling angles φ, ψ it is, according

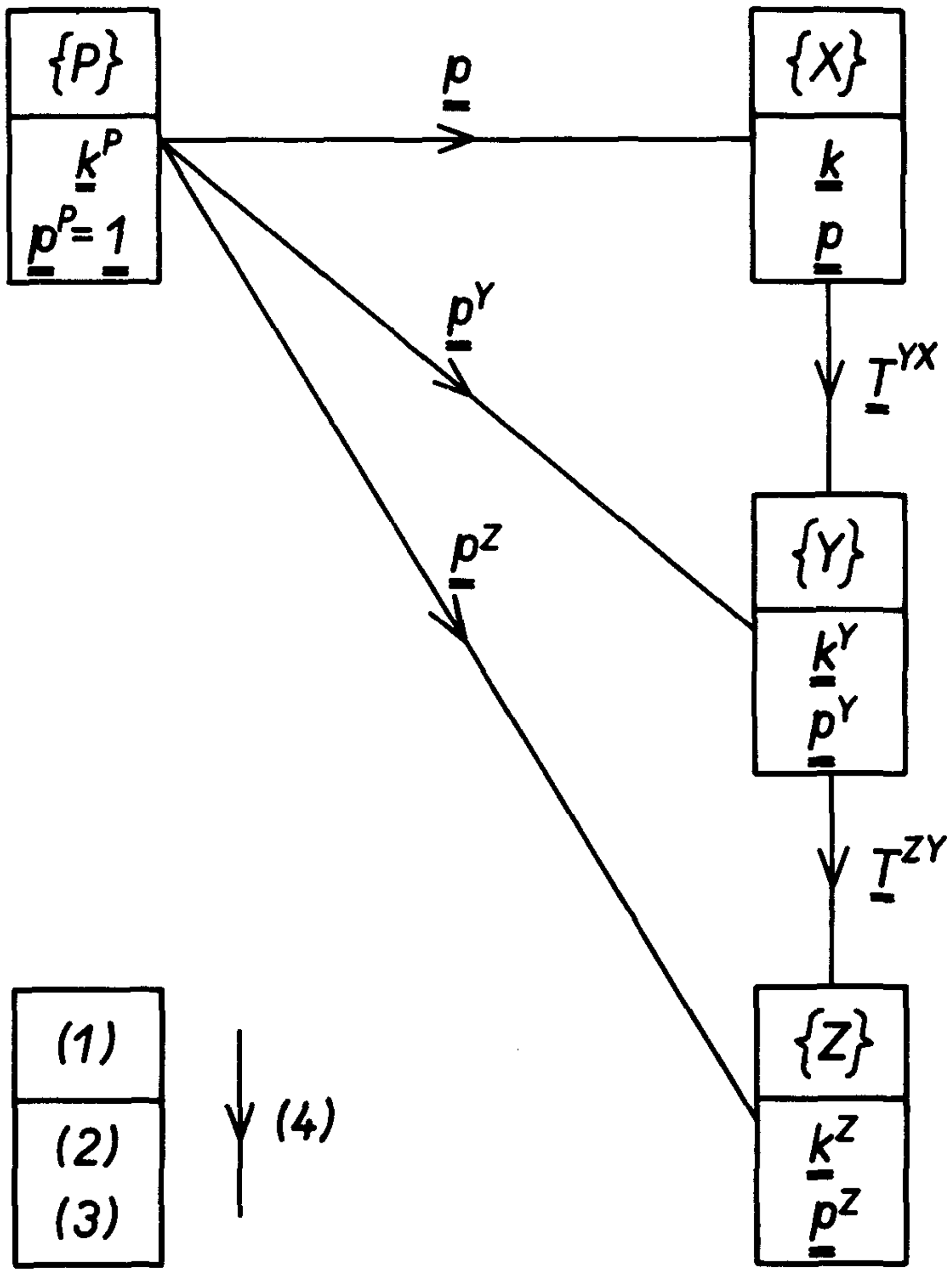

Fig. 3.3 Schematic diagram of the relations between the susceptibility parameters in different coordinate systems. (1) Coordinate system symbol, (2) matrix representing the susceptibility tensor, (3) matrix expressing the principal directions, (4) transformation matrix.

to (3.1), possible to derive for the matrix $\underline{\underline{T}}^{YX}$

(3.6) $\underline{\underline{T}}^{YX}$ =

$\cos\varphi$	$-\sin\varphi\ \cos\psi$	$\sin\varphi\ \sin\psi$
$\sin\varphi$	$\cos\varphi\ \cos\psi$	$-\cos\varphi\ \sin\psi$
0	$\sin\psi$	$\cos\psi$

Similarly, the $\underline{\underline{T}}^{ZY}$ matrix is the transformation matrix of the geographic system $\{Y\}$ to the tectonic system $\{Z\}$. On the basis of the definition of the angles λ , μ , ν we can, according to (3.1), derive for it

(3.7) $\underline{\underline{T}}^{ZY}$ =

$\cos\lambda\sin\nu$ + $\sin\lambda\cos\mu\cos\nu$	$\sin\lambda\sin\nu$ - $\cos\lambda\cos\mu\cos\nu$	$-\sin\mu\ \cos\nu$
$\cos\lambda\cos\nu$ - $\sin\lambda\cos\mu\sin\nu$	$\sin\lambda\cos\nu$ + $\cos\lambda\cos\mu\sin\nu$	$\sin\mu\ \sin\nu$
$\sin\lambda\sin\mu$	$-\cos\lambda\sin\mu$	$\cos\mu$

3.4 ALGORITHM OF TRANSFORMATIONS OF SUSCEPTIBILITY PARAMETERS

In this section we shall describe the procedure by which susceptibility parameters can be expressed in the used coordinate systems, and which is used in the computing program.

We assume that the matrices $\underline{\underline{k}}$, $\underline{\underline{p}}$ and $\underline{\underline{k}}^P$ have already been determined. Further computation can then be divided into ten steps that are given below. However, let us, in the meantime, not consider those marked by an asterisk; their meaning will be explained later.

X1)* $\underline{\underline{p}} \leftarrow \mathscr{D}(\underline{\underline{p}})$

X2) $\underline{\underline{k}} = \underline{\underline{p}}\ \underline{\underline{k}}^P\ \underline{\underline{p}}'$

Y1) Transformation matrix $\underline{\underline{T}}^{YX}$ according to (3.6) is computed

Y2) $\underline{\underline{p}}^Y = \underline{\underline{T}}^{YX}\ \underline{\underline{p}}$

Y3)* $\underline{\underline{p}}^Y \leftarrow \mathscr{D}(\underline{\underline{p}}^Y)$

Y4) $\underline{\underline{k}}^Y = \underline{\underline{p}}^Y\ \underline{\underline{k}}^P\ (\underline{\underline{p}}^Y)'$

Z1) Transformation matrix $\underline{\underline{T}}^{ZY}$ according to (3.7) is computed

Z2) $\underline{\underline{p}}^Z = \underline{\underline{T}}^{ZY}\ \underline{\underline{p}}^Y$

Z3)* $\underline{\underline{p}}^Z \leftarrow \mathscr{D}(\underline{\underline{p}}^Z)$

Z4) $\underline{\underline{k}}^Z = \underline{\underline{p}}^Z\ \underline{\underline{k}}^P\ (\underline{\underline{p}}^Z)'$

In the point X2) tensor $\underline{\underline{k}}$ is computed although it has been computed earlier in a different way. The new computation serves to check the correctness and numerical accuracy of the principal directions and principal susceptibilities found. Transformation equations in the steps Y2) and Z2) immediately follow from (3.2); one must only realize that it is a matter of a simultaneous transformation of a triad of vectors.

It now remains to explain the steps X1), Y3), and Z3), that are marked by an asterisk.

Principal susceptibility directions are - strictly speaking - determined by non-oriented straightlines. Hence it follows that the sense of the vectors $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$ is not defined a priori. We shall get it more or less accidentally according to the numerical method used, and it is arranged subsequently according to certain additional requirements.

Let us first notice the situation in the specimen system. We claim the vectors $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$ to form, after an arrangement, with the positive directions of the axis x_3 angles from the interval $\langle 0^\circ, 90^\circ \rangle$. This is how all the endpoints of these vectors get on the hemisphere S_3 whose pole is defined by the positive direction of the axis x_3. As we imagine that the axis x_3 aims vertically downwards, we speak about S_3 as about the "lower" hemisphere.

The required arrangement will be formally written as follows

$$(3.8) \qquad \underline{p}_j \longleftarrow \underline{p}_j \operatorname{sign}(p_{3j}) \ .$$

The corresponding arrangement of the matrix $\underline{\underline{p}}$ then is

$$(3.9) \qquad \underline{\underline{p}} \longleftarrow \left[\underline{p}_1 \operatorname{sign}(p_{31}) \quad \underline{p}_2 \operatorname{sign}(p_{32}) \quad \underline{p}_3 \operatorname{sign}(p_{33}) \right] ,$$

that will be shortened to

$$(3.10) \qquad \underline{\underline{p}} \longleftarrow \mathscr{D}(\underline{\underline{p}}) \ .$$

The arranged matrix $\underline{\underline{p}}$ has altogether non-negative elements in its third row. It is not difficult to show, that this arrangement will not interfere with the correctness of the transformation equation in the step X2), neither with the correctness of further transformations.

This expleins the meaning of the step X1). Also in the geographic and tectonic systems analogous demands are made on the vectors of principal directions. That is why steps Y3) and Z3) are inserted.

3.5 SPHERICAL COORDINATES, LAMBERT PROJECTION

For interpretation purposes the vectors of principal directions (sometimes also other vectors) are expressed in spherical coordinates related either to the system of specimen, or to the geographic or tectonic system. We shall limit ourselves only to the system of specimen, in the remaining systems the situation is analogous.

As the vectors of principal directions are unit vectors, only two coordinates may be considered - declination and inclination. The projection of the vector $\underline{p}_j$ to the plane $(x_1\ x_2)$, will be denoted $\underline{p}_j^*$. By declination D_j of the direction $\underline{p}_j$ is understood the angle between the positive direction of the axis x_1 and the vector $\underline{p}_j^*$ measured in the positive sense of revolution. By inclination I_j of the direction in question we mean the angle that is formed by $\underline{p}_j$ and the plane $(x_1\ x_2)$; generally, $I_j \in (-90^\circ, +90^\circ\rangle$. Nevertheless, we shall assume that the sense of the $\underline{p}_j$ vector was chosen in the way stated in the preceding section. Then the range of the inclination narrows, $I_j \in \langle 0^\circ, 90^\circ\rangle$.

Declination and inclination serve to represent, by means of a suitable projection, principal directions in the plane. The most frequently used Lambert equal-area projection should be mentioned here.

Lambert projection maps the unit hemisphere onto a circle of the radius $\sqrt{2}$. Practical reasons will make us modify the scale factor so that the circle will have a unit radius. We shall concretely consider the projection of the "lower" hemisphere, i.e. the hemisphere whose pole is defined by the positive direction of the x_3 axis, see also section 3.4. The pole will be projected as the centre of the circle and will represent the origin of polar coordinates in the plane. The endpoint of the vector $\underline{p}_j$ will be projected as a point whose polar coordinates will be R_j and D_j; the radial coordinate R_j is given by the equation

$$R_j = (1 - \sin I_j)^{\frac{1}{2}} \quad . \tag{3.11}$$

Generally, nets of Lambert projection are used, and the direction considered is plotted on the basis of declination D_j and inclination I_j. It is, however, of greater advantage to compute the radial coordinate R_j and plot D_j and R_j. In that case, no net is necessary, greater accuracy is achieved, and there is no difficulty in changing the scale of projection. That is why the determination of the radial coordinate of Lambert projection was included into the computing program.

4. PROGRAM ANISO 10

A computing program has been developed to make it easy to apply theoretical knowledge stated in the preceding chapters to the practical problems of anisotropy investigation. The program is called ANISO 10 and is written in FORTRAN IV language. It processes the data measured by the bridge set KAPPABRIDGE KLY 1. There would be, nevertheless, no difficulty in adapting the program for another instrument based on a similar principle.

From a system of 15 directional susceptibilities measured in a rotatable system of directions, the program computes the components of the susceptibility tensor, principal susceptibilities and principal directions, anisotropy factors, statistical estimates of accuracy of the results obtained, and statistics for anisotropy testing. Tensor components and principal directions can be expressed in three coordinate systems, namely in the system of specimen, geographic system and tectonic system. In principle, the computation proceeds according to the equations given in chapter 1 to 3. The determination of the principal susceptibilities and principal directions, i.e. the determination of eigenvalues and eigenvectors of the matrix expressing susceptibility tensor, is the most difficult numerical task. This task is solved by the iterative method of Jacobi [2]. The algorithm was taken over, after some slight modifications, from source [9], where it is described in the ALGOL language. The algorithm is highly effective; a quick convergence of the process and the orthogonality of the principal directions found are ensured, even in difficult cases, when principal susceptibilities almost coincide.

The results are printed on a line printer (LP). As far as required, the most important results are output on punch cards. In one run the program can successively process results of measurements on an arbitrary number of specimens. The correctness of the input data deck is checked to a certain extent, as will be explained in section 4.2.

The output on the cards was introduced to make it possible later to form groups of susceptibility tensors and process them statistically. The fundamental significance of such an extension of the anisotropy study certainly need not be emphasized.

In this chapter fundamental informations about the ANISO 10 program, important from the view of its practical use, will be given.

To facilitate the orientation in the program listing, names in brackets are given in the description of I/O data in sections 4.1. and 4.2, by which the corresponding variables are designated in the program.

The listing of the ANISO 10 program is not a part of this work; it can be obtained from Geofyzika, n.p.

4.1 INPUT DATA

4.1.1 Introductory notes

Until now we have assumed to obtain by measuring a system of directional susceptibilities $\tilde{\varkappa}_{Di}$ (i = 1, 2, ..., 15). But the device KLY 1 does not directly give the directional susceptibilities; its data depend not only on the directional susceptibility of the specimen, but also on other factors, especially on the volume of the specimen and of the susceptibility of the specimen holder. When measuring in the lowest ranges, it is also the temperature drift of the parameters of the measuring coils that influences the data of the device. The practical situation is such that, after a simple processing of the primary data, we get a system of quantities $B^{(i)}$ ($i = 1, 2, \ldots, 15$), which are essentially the readings of the principal measuring potenciometer corrected, if necessary, for the temperature drift [1]).

From the quantities $B^{(i)}$ the directional susceptibilities can be computed according to the equation

$$\tilde{\varkappa}_{Di} = (KB^{(i)} - \vartheta_H)\, C_{VOL} , \tag{4.1}$$

where K is the range factor of the respective measuring range, ϑ_H is the correction for the susceptibility of the holder, C_{VOL} the correction for specimen volume [2]). The correction C_{VOL} is given by the relation

$$C_{VOL} = V_0/V , \tag{4.2}$$

where V_0 is the nominal specimen volume, for which the device has been calibrated (8 cm^3), V the actual volume of the specimen.

----

[1]) The description of the measuring process proper and of the preliminary processing data is not the subject of this work and is stated in the instruction manual [14].

[2]) In the instruction manual [14] - with regard to its bearing - a more extensive terminology is used. The expression given on the right side of the equation (4.1) in paranthesis is called there directional total susceptibility of the specimen, similarly the quantity ϑ_H is called total susceptibility of the holder.

4.1.2 Survey of input data, form

Now we can already present a complete specification of the input data; it will be suitable to do so in connection with the form for measurement recording. Examples of filled up forms are shown in figs. 4.1 and 4.2.

The input data to be punched on the cards are contained in the upper part of the form, that consists of 5 lines; from each line one card arises. In the horizontal direction the upper part of the form is divided into 7 "blocks" marked by numbers 1, 3, 20, 30, ..., 60 that give the positions on the card from which the corresponding data are to be punched.

BLOCK 1

Lines 1 to 5 successively contain the numbers 1, 2, 3, 4, 5 [N], which serve to serial numbering of the cards for one specimen.

BLOCK 3

Line 1 denotation of specimen [NØM, NØ]. Chain of 12 alpha-numeric characters, the blank being understood as a character.

BLOCK 20

Line 1 three asterisks (* * *). Of sense only as an orientation aid in punching.

Line 2 program-modification switch [SWT(3)]. It is formed by a triad of digits that determine the coordinate system in which the computation, print and punching are to be run. The digits successively relate to the coordinate system of the specimen, the geographic and tectonic coordinate systems. If in the given system computation and print on the LP (computation, print on the LP and card punching) are required, digit 1 (2) is written in the respective place. If no computation is required, 0 is written. E.g. the digits 120 mean that in the system of the specimen computation and print on the LP will run, in the geographic system a card will be punched in addition, in the tectonic system no computation will be carried out. If 0 occurs in the triad of digits followed by at least one non-zero digit, the program will "subsitute" this zero by the digit 1. E.g. the combination 102 will be processed as 112, combination 001 as 111.

Line 3 correction for the susceptibility of the holder ϑ_H [THH], see section 4.1.

Line 4 correction for the specimen volume C_{VOL} [CV], see section 4.1.

Line 5 range factor K of the used measuring range [RFK], see section 4.1.

BLOCK 30

Lines 1, 2 sampling angles φ, ψ [PHI, PSI].

Lines 3 to 5 angles λ, μ, ν [LD, MU, NU] determining the orientation of the tectonic system. If the tectonic system is not defined, these lines are not filled up.

BLOCK 40

Lines 1 to 5 quantities $B^{(1)}$ to $B^{(5)}$ [B(1) to B(5)], see section 4.1.

BLOCK 50

Lines 1 to 5 quantities $B^{(6)}$ to $B^{(10)}$ [B(6) to B(10)].

BLOCK 60

Lines 1 to 5 quantities $B^{(11)}$ to $B^{(15)}$ [B(11) to B(15)].

This essentially exhausts the input-data survey and, in the meantime, also the description of the upper part of the form.

Only a few informative notes on the lower part of the form. To the recording of the primary measured values (reading of the potenciometers of the device) the right lower part of the form is designated. This part contains 15 blocks with 5 lines each. Each of the blocks corresponds to one measuring direction, i.e. to one measuring position of the specimen. The measuring positions are marked by arrows, see also the scheme of measuring positions in fig. 1.3. From the primary data measured in the i-th position the quantity $B^{(i)}$ is then determined and written in the respective block and line in the upper part of the form.

In the left lower part of the form there is room for various auxiliary data, for the petrographic description of the specimen, etc.

1	3	20	30	40	50	60
1	NJ_14/3/2	***	60.	1069.8	1054.2	1072.
2		120	85.	1062.3	1082.8	1078.6
3		-2.4	.	1073.8	1061.3	1076.4
4		1.	.	1070.9	1055.9	1072.
5		.1	.	1063.6	1083.1	1078.2

SAMPLE

*** φ
SWITCH ψ
CORR.FOR HOLDER λ
CORR.FOR VOLUME μ
RANGE FACTOR ν

↗	59		⇗	2		⬈	51	
	1073			1055			1073	
	65			5			53	5
	64			7	5		54	
	70			10			56	
↘			⇘			⬊		
	1061	5		1085			1079	
	69			17			59	
	75			18	5		60	
	77			22			60	
↓	80		⇓	28		⬇		
	1075	5		1062			1076	5
	84			30	5		62	
	84			32	5		64	
	87			34	5		64	5
↙			⇙			⬋		
	1072	5		1057	5		1072	
	90	5		38			68	
	91	5		39	5		70	
	95			43	5		69	
↖	99		⇖	42	5	⬉		
	1062	5		1084	5		1080	
	99			45	5		74	
	2	5		45	5		75	
	1			48			78	5

MEASURED BY	COMPUTED BY	DATE
		14/2/72

Fig. 4. 1

Form for recording the anisotropy measurement, specimen NJ 14/3/2.

1	3	20	30	40	50	60
1	CM␣1/1/3	***	68.	1568.	1084.	1190.
2		122	81.	1280.	1387.5	1399.
3		-2.3	210.	1476.	1377.	1111.5
4		1.	65.	1570.5	1090.	1183.5
5		10.	0.	1288.5	1387.	1397.5

SAMPLE

***	φ
SWITCH	ψ
CORR.FOR HOLDER	λ
CORR.FOR VOLUME	μ
RANGE FACTOR	ν

MEASURED BY	COMPUTED BY	DATE
		14/2/72

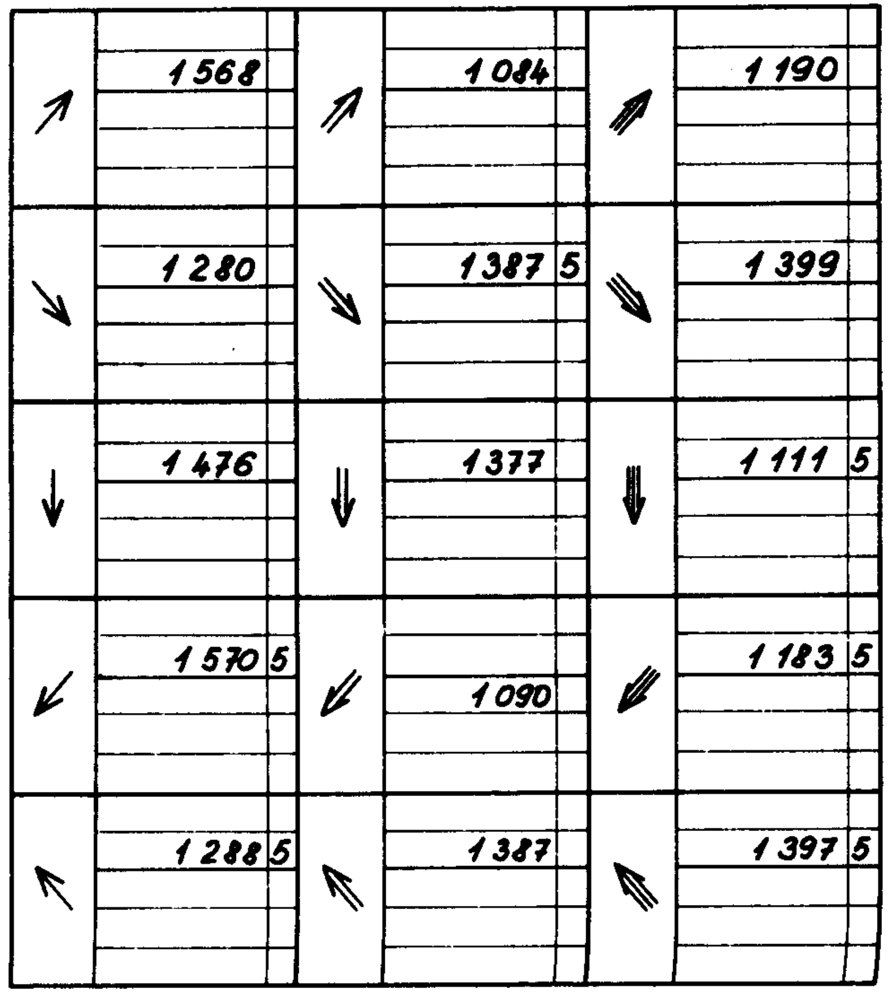

Fig. 4. 2

Form for recording the anisotropy measurement, specimen CM 1/1/3.

```
PROGRAM  ANIS9 10          INPUT AND OUTPUT DATA DECKS (EXAMPLE)          PAGE   1

....*....1....*....2....*....3....*....4....*....5....*....6....*....7....*....8


                                   INPUT
                                   =====

  (ONE BLANK CARD)
1 NJ 14/3/2          ***        60.        1069.8     1054.2     1072.
2                    120        85.        1062.3     1082.8     1078.6
3                    -2.4                  1073.8     1061.3     1076.4
4                    1.                    1070.9     1055.9     1072.
5                    .1                    1063.6     1083.1     1078.2
  (ONE BLANK CARD)
1 CM 1/1/3           ***        68.        1568.      1084.      1190.
2                    122        81.        1280.      1387.5     1399.
3                    -2.3       210.       1476.      1377.      1111.5
4                    1.         65.        1570.5     1090.      1183.5
5                    10.        0.         1288.5     1387.      1397.5
6 END OF DATA


                                   OUTPUT
                                   ======

NJ 14/3/2        109.43    .00065  1.0062 1.0001  .9936  .0012 -.0096  .0104     2
CM 1/1/3       13200.00    .00318   .9189 1.0099 1.0711  .1775 -.1161  .0361     2
CM 1/1/3       13200.00    .00318  1.1195 1.0953  .7851 -.0785 -.1048 -.0148     3
```

Fig. 4. 3 Input and output card decks, specimens NJ 14/3/2 and CM 1/1/3.

4.1.3 Punching and composition of input data deck

The group of input data cards for one specimen is formed by six cards. It begins by a blank (zero) card, then five cards follow, punched according to the form for measurement recording, see preceding par. The characters written in a certain line in the individual fields are punched from the marked positions (1, 3, 20, 30, ..., 60), closely following each other without blanks, as far as the blanks are not especially marked.

The input data deck for n specimens (n = 1, 2, ...) consists of card groups for the respective specimens ranged one after the other and from the closing, sentinel card

6 END ØF DATA

that stops the computation.

In the upper part of fig. 4.3 we can find an example of input data decks for both specimens selected the measuring of which is recorded in figs. 4.1 and 4.2.

4.2 OUTPUT DATA

If the input data deck is composed properly, computation for all the specimens will run successively, the results are printed on the LP, one page for each specimen. If also output on the cards is required, for each specimen 1 to 3 cards, according to the program modification, are punched, cf. par. 4.1.2.

After finishing the whole computation, the message "END ØF DATA" is printed on the LP as a check that the whole input data deck was processed.

The program checks the formal correctness of the composition of the input data deck. If an erroneous group of cards occurs there, all the names of specimens contained in the erroneous group are printed on the LP, and each name is followed by the message "ERRØNEØUS DATA CARD GRØUP - CØMPUTING SUPPRESSED". The suppression of the computation concerns the erroneous group only, the computation proceeds further in the normal way.

4.2.1 Output on the line printer

A demonstration of printing the results on the LP can be seen in figs. 4.4 and 4.5. They are results for specimens the input data of which were given in fig. 4.3.

For each specimen always one page of results is printed. The results have the character of statistical estimates; this circumstance will not be emphasized, see the agreement at the beginning of chapter 3.

On the top of the left side of the page the denotation of the specimen is printed [NØM], that is stressed by underlining with a row of asterisks. Below the denotation of the specimen is the mean susceptibility $\varkappa$ [KPM] in 10^{-6} (SI system). One line lower is in the same units behind the denotation " + - " the standard error $s(\varkappa)$ of the mean susceptibility [SKPMA]. The mean susceptibility is printed rounded-off. The order of the least significant valid place corresponds to the order of the error; on the places of lower orders zeros are printed, if necessary.

If the order of the least significant valid place is negative, the mean susceptibility is printed in the format with two decimal places, otherwise without decimal places. In the right part of the heading the name of the program is given.

Further results are printed in several sections; by section a group of lines is here understood that represents, optically and logically, a certain whole. In the first section all the input data are printed in the same arrangement as in the form for measurement recording. In the right side of this section there is a group of 15 data [DEL(15)], denoted as RESIDUES. These are residual errors of the quantities $B^{(1)}$ to $B^{(15)}$, following from the least square method; they can be also understood as errors of directional susceptibilities expressed in divisions of the scale of the device. Immediatelly below this group there is a datum denoted FUND. ST. ERR. [SB]. It is the fundamental standard error of the measurement which is defined as the standard error of the directional susceptibility; here it is expressed in scale divisions.

The second section begins with the datum NØRMALIZED FUND. ST. ERR., which is the fundamental standard error (i.e. the standard error of the directional susceptibility) in the normalized form [S]. The normalizing factor here, as well as in other cases, is always the mean susceptibility $\varkappa$.Then follow statistics F, F_{12}, F_{23} [F, F12, F23] for anisotropy testing, error parameters s_{T12}, s_{T23}, s_{T13} [ST(3)], confidence angles e_{12}, e_{23}, e_{13} for the significance level 90 % and 95 % [E90(3), E95(3)].

The third section contains the normalized mean susceptibility (that, understandably, equals one and is printed for formal reasons only), principal susceptibilities $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ in the normalized form [KP(1,1), KP(2,2), KP(3,3)] and anisotropy factors H_1 to H_6 [H1 to H6)]. Under the individual data the respective standard errors are given. All the principal susceptibilities have the same standard error.

The fourth section contains the spherical coordinates of principal directions; i.e. declinations D_1, D_2, D_3 [DECL(3)], inclinations I_1, I_2, I_3 [INCL(3)] and radial coordinates R_1, R_2, R_3 of the endpoints

```
NJ 14/3/2                                                                          ANISOTROPY OF SUSCEPTIBILITY (PROGRAM ANISO 10)
************        -5
     109.43 * 10     MEAN SUSC. (SI)
+-       .02

                                   ORIENTATION                 DATA MEASURED (SCALE DIV.)             RESIDUES (SCALE DIV.)

                                PHI          60.0           1069.8    1054.2    1072.0            -.9        -.7        .1
SWITCH                  120     PSI          85.0           1062.3    1082.8    1078.6           -1.0        -.0        .3
CORR. FOR HOLDER      -2.40     LD             .0           1073.8    1061.3    1076.4             .5         .5       -.5
CORR. FOR VOLUME      1.000     MU             .0           1070.9    1055.9    1072.0             .2        1.0        .1
RANGE FACTOR             .1     NU             .0           1063.6    1083.1    1078.2             .3         .3       -.1
                                                                                                 FUND. ST. ERR.          .7

                                                                       ACCURACY OF PRINCIPAL DIRECTIONS

        NORMALIZED            TESTS FOR ANISOTROPY                ERROR PARAMETERS                CONFIDENCE ANGLES,
        FUND. ST. ERR.                                                                            90% AND 95% LEVEL
             S                 F        F12       F23          ST12      ST23      ST13          E12        E23        E13

          .00065            459.95    115.46    505.43         .0329     .0157     .0106         4.6        2.2        1.5
                                                                                                 5.5        2.6        1.8

        NORMALIZED        NORMALIZED PRINCIPAL SUSC.                               ANISOTROPY FACTORS
        MEAN SUSC.
                              (1)       (2)       (3)           H1        H2        H3            H4         H5         H6

           1.0000           1.0134    1.0036     .9830        1.0098    1.0210    1.0310        1.0260     1.0203     1.0111
ST. ERR.    .0002            .0004      ***       ***          .0006     .0007     .0007         .0006      .0006      .0011

                     PRINCIPAL DIR. - SPHERICAL COORD.     PRINCIPAL DIR. COSINES (COLUMNS)          NORMALIZED TENSOR

SPECIMEN          DECL       80.4     346.7     255.5          .0829     .9725    -.2176        1.0027     -.0034      .0029
COORD.            INCL       60.4       2.1      29.6          .4877    -.2300    -.8422                    .9913      .0127
SYSTEM            LMPR       .362      .982      .712          .8691     .0363     .4933                              1.0060

GEOGRAPHIC        DECL      335.8     236.7     128.9          .7544    -.5349    -.3804        1.0062      .0012      .0104
COORD.            INCL       34.2      13.1      52.7         -.3398    -.8141     .4709                   1.0002     -.0096
SYSTEM            LMPR       .662      .880      .452          .5615     .2260     .7960                               .9936

JACOBI ITERATIVE PROCESS CHARACTERISTICS          ACC =   .78E-07                NT =  5
```

Fig. 4.4 Line printer output, specimen NJ 14/3/2.

```
CM 1/1/3                                                            ANISOTROPY OF SUSCEPTIBILITY (PROGRAM ANISO 10)
************      -6
  13200.     * 10    MEAN SUSC. (SI)
+-    10.

                               ORIENTATION                 DATA MEASURED (SCALE DIV.)           RESIDUES (SCALE DIV.)

                                PHI          68.0         1568.0   1084.0   1190.0           -.3       -5.7        3.4
SWITCH                122       PSI          81.0         1280.0   1387.5   1399.0          -3.3       -2.5         .9
CORR. FOR HOLDER    -2.30       LD          210.0         1476.0   1377.0   1111.5          -2.3        3.6        5.1
CORR. FOR VOLUME    1.000       MU           65.0         1570.5   1090.0   1183.5           2.2         .3       -3.1
RANGE FACTOR         10.0       NU             .0         1288.5   1387.0   1397.5           5.2       -3.0        -.6
                                                                                             FUND. ST. ERR.        4.2

                                                                        ACCURACY OF PRINCIPAL DIRECTIONS

      NORMALIZED            TESTS FOR ANISOTROPY             ERROR PARAMETERS               CONFIDENCE ANGLES,
      FUND. ST. ERR.                                                                        90% AND 95% LEVEL
           S                 F        F12       F23       ST12      ST23      ST13          E12        E23        E13

       .00318           4137.25   981.50   4641.48       .0113     .0052     .0036          1.6         .7         .5
                                                                                            1.9         .9         .6

      NORMALIZED      NORMALIZED PRINCIPAL SUSC.                            ANISOTROPY FACTORS
      MEAN SUSC.
                          (1)       (2)       (3)         H1        H2        H3            H4         H5         H6

        1.0000         1.1955    1.0551     .7492       1.1333    1.4084    1.5962        1.5023     1.3255     1.2427
ST. ERR.  .0008         .0020      ***       ***         .0032     .0043     .0044         .0039      .0032      .0062

               PRINCIPAL DIR. - SPHERICAL COORD.    PRINCIPAL DIR. COSINES (COLUMNS)         NORMALIZED TENSOR

SPECIMEN          DECL      325.2     55.3     235.0      .8210     .4792    -.3104        1.1204    -.1080      .0801
COORD.            INCL         .1     32.7      57.3     -.5710     .6913    -.4428                  1.0410      .1138
SYSTEM            LMPR       .999     .678      .399      .0023     .5408     .8412                              .8386

GEOGRAPHIC        DECL      241.6     26.4     139.0     -.3925     .5745    -.7183         .9189      .1775      .0361
COORD.            INCL       34.3     50.1      17.8     -.7269     .2848     .6249                  1.0099     -.1161
SYSTEM            LMPR       .661     .482      .833      .5636     .7674     .3058                             1.0712

TECTONIC          DECL      314.6    221.9      71.5      .6939    -.7126     .1034        1.1195    -.0784     -.0148
COORD.            INCL        8.9     16.7      71.0     -.7033    -.6399     .3096                  1.0954     -.1048
SYSTEM            LMPR       .920     .844      .234      .1545     .2875     .9452                              .7851

JACOBI ITERATIVE PROCESS CHARACTERISTICS          ACC =    .13E-08               NT =   5
```

Fig. 4.5 Line printer output, specimen CM 1/1/3.

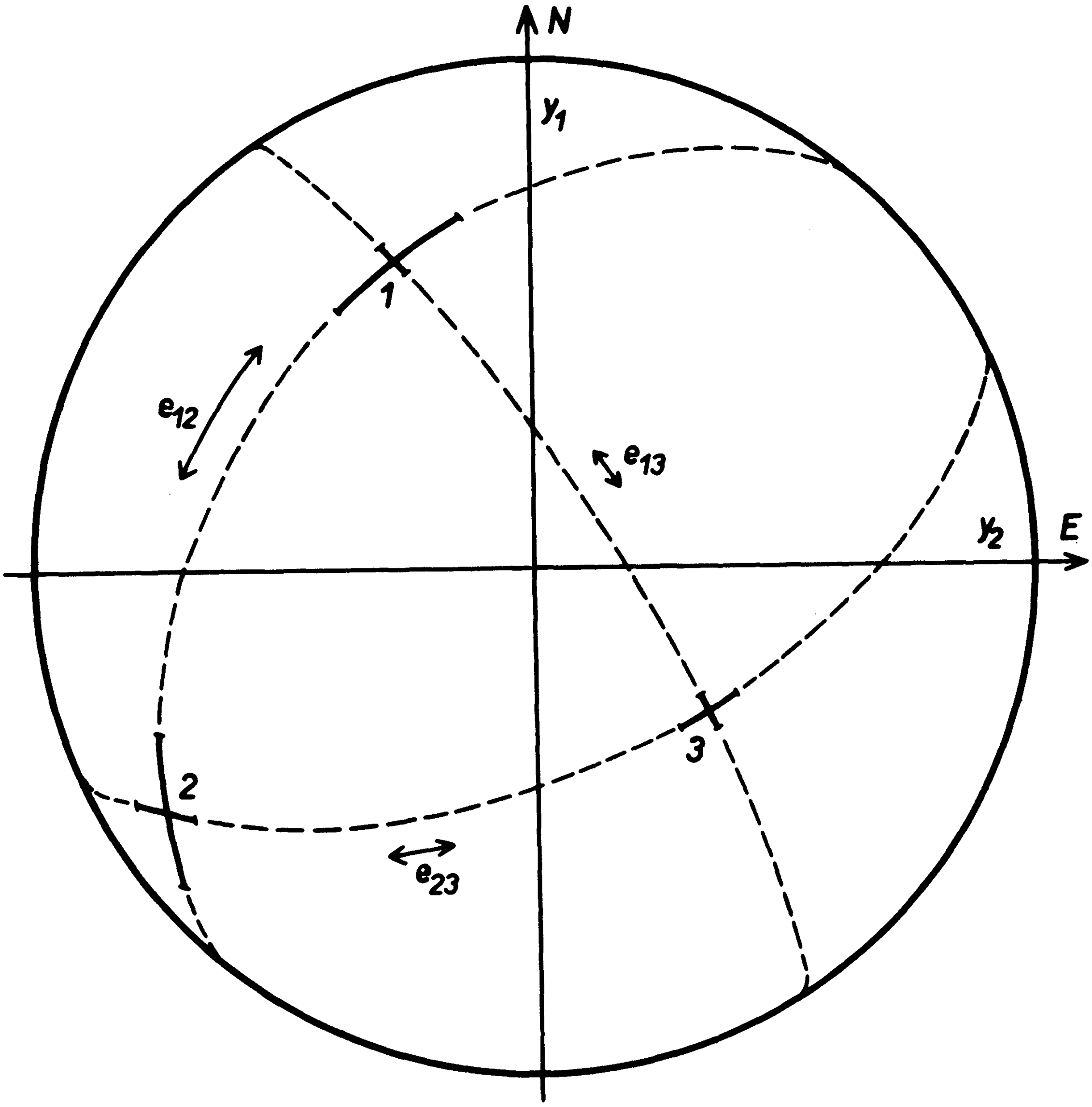

Fig. 4.6 Specimen NJ 14/3/2, geographic coordinate system, principal directions and 95 % confidence angles in Lambert projection; the confidence angles are twice magnified.

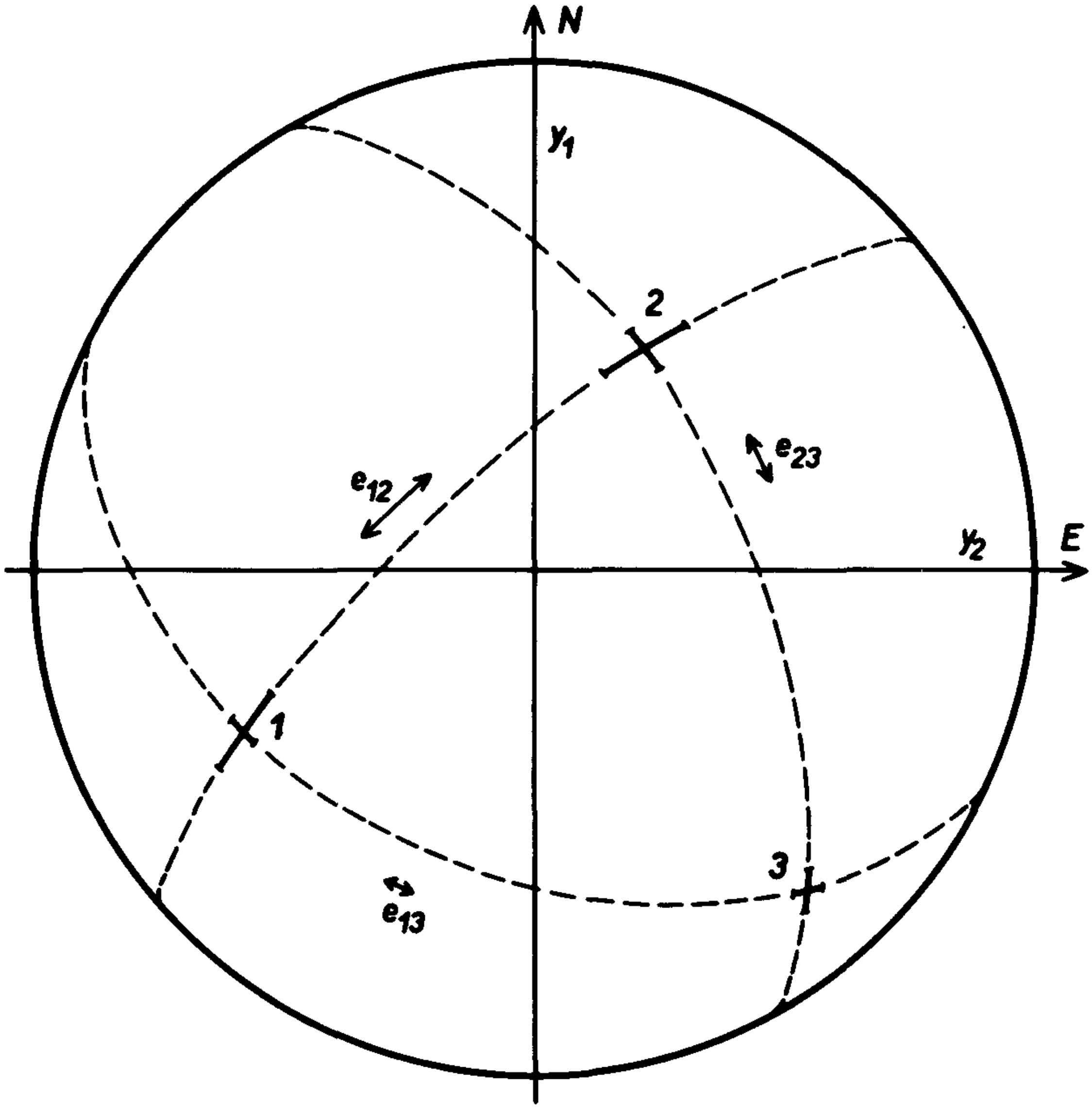

Fig. 4. 7 Specimen CM 1/1/2, geographic coordinate system, principal directions and 95 % confidence angles in Lambert projection; the confidence angles are four times magnified.

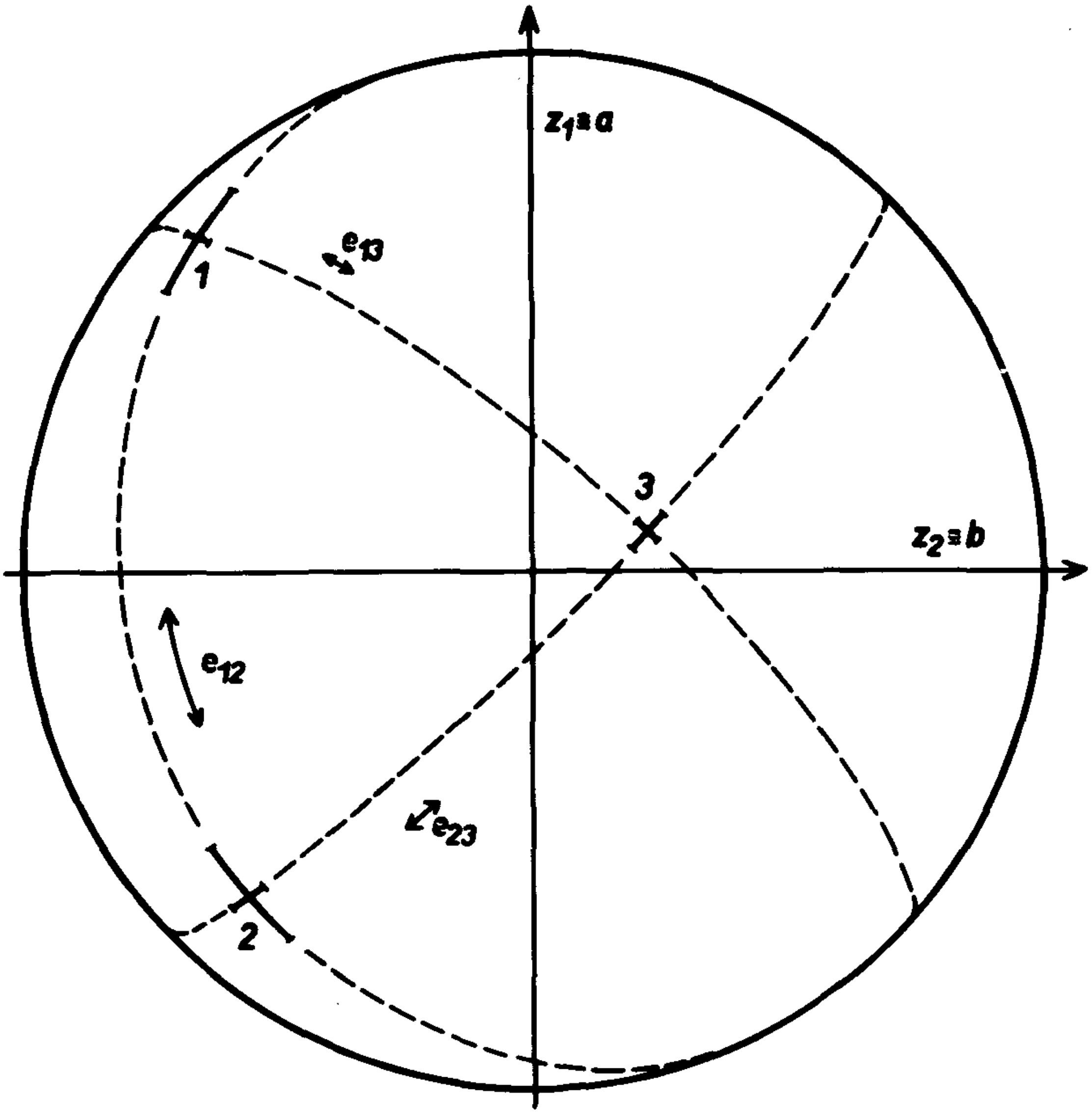

Fig. 4. 8 Specimen CM 1 / 1 / 3, tectonic coordinate system, principal directions and 95 % confidence angles in Lambert projection; the confidence angles are four times magnified.

of the vectors of principal directions in the Lambert projection [LMPR(3)]. (The principal directions and their confidence angles for the specimens considered are plotted in figs. 4.6 to 4.8.) Further,the section contains directional cosines of the principal directions [P(3,3)] and the components of susceptibility tensor $\underline{\underline{k}}$ in the normalized form [KB(3,3)]. These data are printed in all coordinate systems required.

In the last, separate line there are data about the cours of Jacobi iterative process. ACC gives a certain characterization of its accuracy, NT states the number of iterations (of elementary transformations); details can be found in the program listing. Typical ACC values are of the order 10^{-6}, NT usually is 5 to 8.

4.2.2 Output on punch cards

If required,the principal results are punched on cards. An illustration of output cards for both specimens chosen can be found in the lower part of fig. 4.3.

On the card the following data are successively punched : denotation of specimen [here NØ], mean susceptibility $\varkappa$ [KPM], fundamental standard error s in the normalized form [S], components of susceptibility tensor $\underline{\underline{k}}$ in the normalized form [KB(1,1), KB(2,2), KB(3,3), KB(1,2), KB(2,3), KB(1,3)], number of coordinate system [NSW]; this number for the system of the specimen (geographic, tectonic) is 1 (2, 3).

From the output cards it is possible to compose decks for a further program denoted ANS21, that statistically processes the results measured on a group of specimens coming from a certain locality or a certain geological body. ANS21 makes it possible to characterize the anisotropy of magnetic susceptibility of such an object as a whole.

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